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Coisotropic variational problems

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Abstract

In this paper we study constrained variational problems in one independent variable defined on the space of integral curves of a Frenet system in a homogeneous space G/H. We prove that if the Lagrangian is G-invariant and coisotropic then the extremal curves can be found by quadratures. Our proof is constructive and relies on the reduction theory for coisotropic optimal control problems. This gives a unified explanation of the integrability of several classical variational problems such as the total squared curvature functional, the projective, conformal and pseudo-conformal arc-length functionals, the Delaunay and the Poincaré variational problems.

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1. Introduction

The present paper is an outcome of our attempt to understand the general mechanisms underlying the integrability of constrained variational problems for curves of constant type in homogeneous spaces [10,11,17,18]. The Pfaffian differential systems arising from curves of constant type lead to the notion of *generalized Frenet system* for curves of a homogeneous space G/H. Roughly speaking, a generalized Frenet system of order k on G/H is a G-invariant submanifold $S \subset J^k(\mathbb{R}, G/H)$ of the jet space $J^k(\mathbb{R}, G/H)$, which may be linearized by a left-invariant affine sub-bundle of $T(G)$. From the geometrical viewpoint the integral curves of such systems are canonical lifts of curves of constant type on G/H.

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The most elementary example is the classical Frenet–Serret differential system for generic curves in Euclidean space. We then consider a G-invariant Lagrangian and we investigate the corresponding Euler–Lagrange system. The general construction of the momentum space and of the Euler–Lagrange system of a constrained variational problem in one independent variable is due to Griffiths (we refer to [5,12,16] as the standard references on the subject and to [6] as the original source of inspiration of the approach developed by Griffiths). We adhere to the terminology introduced in [5,12] and say that a Lagrangian L is *non-degenerate* if the *momentum space* Y is odd-dimensional and if the canonical 2-form on Y has maximal rank. We prove that if the Lagrangian L is G-invariant and *coisotropic* (see Definition 4.3) then the extremal curves of the variational problem can be found by quadratures. The proof relies on the reduction theory of Hamiltonian systems with symmetries (see [2,9,14,15] for the standard theory in the symplectic category and [1,5,24,27] for generalizations to contact geometry, time-dependent Hamiltonian systems and Poisson manifolds). One of the ingredients of the proof is a concrete geometric description of the Marsden–Weinstein reduced spaces in terms of the *phase portraits* of the system. This procedure is constructive and applies to several concrete examples (see Refs. [5,7,12,16,20–23]).

The paper is organized as follows. In the next section we recall the basic definitions and properties of linear control systems on Lie groups and Frenet systems of curves in homogeneous spaces. In Section 3, we examine variational problems defined by invariant Lagrangians for linear control systems on Lie groups. From a geometrical viewpoint we deal with kth order variational problems for curves of constant type in a homogeneous space that depend on the generalized curvatures. Since all the derived systems have constant rank, the extremal curves of the variational problem are the projections of the integral curves of the Euler–Lagrange system. Therefore, we focus our attention on the momentum space and on the Euler–Lagrange system. First we investigate the geometry of the momentum space Y of a *regular invariant Lagrangian* of a linear control system of a Lie group G. We show that Y is of the form $G \times \mathcal{F}$, where $\mathcal{F}$ is an immersed submanifold of $\mathfrak{g} \times \mathfrak{g}^*$ (we call $\mathcal{F}$ the *phase space* of the variational problem). Next we study non-degenerate Lagrangians. We prove that if L is non-degenerate, then the phase space $\mathcal{F}$ can be realized as a submanifold of $\mathfrak{g}^*$. We define the linearized phase portraits and the Legendre transform and analyze the structure of the characteristic vector field of a non-degenerate Lagrangian. In Section 4 we study *coisotropic Lagrangians*. We prove that the integral curves of the characteristic vector field passing through a point of the bifurcation set are orbits of one-parameter subgroups of the symmetry group G. Therefore, from this point on, we focus our attention on the regular part Y_{r} of the momentum space. We show that Y_{r} is of the form $G \times \mathcal{F}_{\mathrm{r}}$, where $\mathcal{F}_{\mathrm{r}}$ is an open subset of the phase space. We prove that $\mathcal{F}_{\mathrm{r}}$ intersects the coadjoint orbits, $\mathcal{O}(\mu)$, of G transversally and that $\mathcal{P}_{\mathrm{r}}(\mu) = \mathcal{F}_{\mathrm{r}} \cap \mathcal{O}(\mu)$ are smooth curves (referred to as the *phase portraits*). Subsequently we introduce the moment map $J : Y_{\mathrm{r}} \to \mathfrak{g}^*$ and prove that the Marsden–Weinstein reduction $J^{-1}(\mu)/G_\mu$ can be naturally identified with the phase portrait $\mathcal{P}_{\mathrm{r}}(\mu)$. We also show that every $\mu \in J(Y_{\mathrm{r}})$ is a regular element of $\mathfrak{g}^*$ which implies that the isotropy subgroups G_μ are Abelian, for every $\mu \in J(Y_{\mathrm{r}})$. We then examine more closely the phase flow ϕ and the characteristic vector field ξ. We prove that if the Lie algebra $\mathfrak{g}$ possesses a non-degenerate Ad-invariant inner product then the differential equation fulfilled by the phase flow can be written in Lax form. From the Noether conservation theorem we know that the characteristic vector field ξ is tangent to the fibers $J^{-1}(\mu)$ of the moment map. We define a canonical connection form θ^{μ} on the Marsden–Weinstein fibrations $J_{\mathrm{r}}^{-1}(\mu) \to \mathcal{P}(\mu)$ whose horizontal curves are the integral curves of the characteristic vector field. Since the base is one-dimensional and the structure group G_{μ} is Abelian, the horizontal curves can be found by a single quadrature. This shows that the extremal curves of an invariant coisotropic Lagrangian are integrable by quadratures. As a byproduct, we prove that if the canonical connection θ_{μ} is complete, which is generically the case when μ is a regular value of the moment map, then the connected components of $J_{\mathrm{r}}^{-1}(\mu)$ are Euclidean cylinders and the characteristic vector field ξ can be linearized on $J_{\mathrm{r}}^{-1}(\mu)$. We would like to stress that the connection form θ^{μ} can be constructed explicitly from the data of the problem, so that the integration process can be performed in a completely explicit way.

Finally, in two appendices, we summarize the background material that we use from the theory of Pfaffian differential systems and constrained variational problems in one independent variable.

Throughout the paper, we demonstrate how our general results apply to the specific example of isotropic curves in $\mathbb{R}^{(2,1)}$. We show how to derive the Frenet system for such curves, and show that the variational problem is coisotropic if we take the Lagrangian to be a linear function of the curvature. We prove that the phase portraits may be parameterized in terms of elliptic functions, and construct the sections of the Marsden–Weinstein fibration required to reduce the integration to quadratures. Other concrete geometrical examples where the general scheme described in this paper are implemented may be found in [5,6,20–23]. In all of these cases the generic phase portrait is an elliptic curve, so that the extremal curves can be integrated in terms of elliptic functions and elliptic integrals.

2. Linear control systems on Lie groups and Frenet systems in homogeneous spaces

2.1. Linear control systems on Lie groups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. The natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ will be denoted by $(\eta, V) \in \mathfrak{g}^* \times \mathfrak{g} \to \langle \eta; V\rangle \in \mathbb{R}$. We let $\Theta \in \mathfrak{g}^* \otimes \mathfrak{g}$ be the Maurer–Cartan form of G. If we fix a basis $(e_0, \dots, e_n)$ of $\mathfrak{g}$, then $\Theta = \theta^J \otimes e_J$, where $(\theta^0, \dots, \theta^n)$ is the basis of $\mathfrak{g}^*$ dual to $(e_0, \dots, e_n)$.

Definition 2.1. Let $\mathbb{A} \subset \mathfrak{g}$ denote an affine subspace of $\mathfrak{g}$ of the form $P + \mathfrak{a} = \{P + A : A \in \mathfrak{a}\}$, where $P \in \mathfrak{g}$ and $\mathfrak{a} \in Gr_h(\mathfrak{g})$ with $P \notin \mathfrak{a}$. The set of such affine subspaces of $\mathfrak{g}$ will be denoted as $P^h(\mathfrak{g})$. We call $M := G \times \mathbb{A}$ the *configuration space* of the affine subspace $\mathbb{A}$, and denote by $\pi_G : M \to G$ and $\pi_{\mathbb{A}} : M \to \mathbb{A}$ the natural projections onto the two factors.

We now fix a left-invariant form $\omega \in \mathfrak{g}^*$ such that $\langle \omega; P\rangle = 1$ and $\omega \in \mathfrak{a}^{\perp}$ (i.e. $\langle \omega; A\rangle = 0$, for all $A \in \mathfrak{a}$). We may fix a basis $(e_0, \dots, e_n)$ of $\mathfrak{g}$ such that

$$P = e_0, \qquad \mathfrak{a} = \mathrm{span}(e_1, \dots, e_h),$$

and we let $\theta^0, \dots, \theta^n$ be the components of the Maurer–Cartan form with respect to $(e_0, \dots, e_n)$. Such a basis may be chosen so that $\theta^0 = \omega$. Using the projection π_G, we may pull-back the differential 1-forms $\omega, \theta^1, \dots, \theta^n$ to M, to define a set of 1-forms on M which, by the standard abuse of notation, we again denote by $\omega, \theta^1, \dots, \theta^n \in \Omega^1(M)$.[1] Let $k^1, \dots, k^h$ denote the affine coordinates on $\mathbb{A}$ defined by the affine frame $(P, e_1, \dots, e_h)$. We then define the 1-forms

$$\eta^j := \begin{cases} \theta^j - k^j\omega, & j = 1, \dots, h, \\ \theta^j, & j = h+1, \dots, n. \end{cases}$$

We then define the Pfaffian differential system $(\mathcal{A}, \omega)$ on M to be the Pfaffian differential ideal generated by the 1-forms $\{\eta^j : j = 1, \dots, n\}$ with the independence condition given by ω.[2]

Definition 2.2. $(\mathcal{A}, \omega)$ is the linear control system associated to $\mathbb{A} \in P^h(\mathfrak{g})$.

Note that the ideal $\mathcal{A}$ has constant rank, being generated by a rank n sub-bundle, $Z \subset T^*(M)$. The sub-bundle Z is of the form $G \times \mathcal{Z}$, where

$$\mathcal{Z} = \{(Q, \eta) \in \mathbb{A} \times \mathfrak{g}^* : \langle \eta; Q \rangle = 0\} \subset \mathbb{A} \times \mathfrak{g}^*,$$

and where the embedding of Z as a sub-bundle of $T^*(M)$ is given by

$$(g, Q, \eta) \in G \times \mathcal{Z} \to \pi_G^*(\eta)|_{(g,Q)} \in T^*_{(g,Q)}M.$$

We may use the left-invariant trivialization to identify $T(G)$ and $G \times \mathfrak{g}$. The tangent space to M at (g, Q) is then identified with $\mathfrak{g} \oplus \mathfrak{a}$. With this identification at hand, the integral elements of $(\mathcal{A}, \omega)$ at (g, Q) are the one-dimensional subspaces of $\mathfrak{g} \oplus \mathfrak{a}$ of the form (Q, v), where $v \in \mathfrak{a}$.

A smooth curve $\gamma = (\alpha, \beta) : (a, b) \to M$, where $(a, b) \subseteq \mathbb{R}$ is a parameterized integral curve of the control system $(\mathcal{A}, \omega)$ if and only if $\alpha : (a, b) \to G$ is a solution of the linear system $\alpha(t)^{-1}\alpha'(t) = \beta(t)$. Thus, as a control system, the points of the affine space $\mathbb{A}$ play the role of the inputs. Note that if we assign a smooth map $\beta : (-\epsilon, \epsilon) \to \mathbb{A}$ and a point $g_0 \in G$, then there exists a unique integral curve of the control system, $\gamma = (\alpha, \beta)$, satisfying the initial condition $\alpha(0) = g_0$.

Consider the linear subspaces $\mathfrak{a}_k \subset \mathfrak{g}$ defined recursively by

$$\mathfrak{a}_1 = \mathfrak{a} + \mathrm{span}(P), \mathfrak{a}_2 = \mathfrak{a}_1 + [\mathfrak{a}_1, \mathfrak{a}_1], \dots, \mathfrak{a}_k = \mathfrak{a}_{k-1} + [\mathfrak{a}_{k-1}, \mathfrak{a}_{k-1}].$$

The smallest integer N such that $\mathfrak{a}_N = \mathfrak{a}_{N+1}$ is called the *derived length* of $\mathbb{A}$. Note that $\mathcal{Z}_s := \mathbb{A} \times \mathfrak{a}_s^\perp$ is contained in $\mathcal{Z}$, for $s = 1, \dots, N$. We set $Z_s = M \times \mathcal{Z}_s$, $s = 1, \dots, N$

[1] We will generally follow the usual practice in the method of moving frames and omit the pull-back signs to simplify notation. This should cause no confusion as we will clearly specify the manifolds that we are working on.

[2] More invariantly, if we fix $\omega \in \mathfrak{g}^*$ with $\omega \in \mathfrak{a}^\perp$ and $\langle \omega; P \rangle = 1$, then we define the $\mathfrak{g}$-valued 1-form $\hat{\Theta} \in \Omega^1(M, \mathfrak{g})$ by the formula $\hat{\Theta}|_{(g,Q)} := \pi_G^*(\Theta - Q\omega)|_{(g,Q)}$. $\mathcal{A}$ is then the differential ideal generated by $\{\langle \mu; \hat{\Theta} \rangle : \mu \in \mathfrak{g}^*\}$.

and we consider the sequence of sub-bundles

$$Z_N \subset Z_{N-1} \subset \cdots \subset Z_1 \subset Z.$$

If we denote by $\mathcal{A}_s$ the Pfaffian differential ideal generated by Z_s, then

$$\mathcal{A}_N \subset \mathcal{A}_{N-1} \subset \cdots \subset \mathcal{A}_1 \subset \mathcal{A}$$

is the derived flag of the control system (see Refs. [4,12] for more details about derived flags). We have thus proved the following proposition.

Proposition 2.3. *All the derived systems of a linear control system on a Lie group G have constant rank.*

2.2. Frenet systems in homogeneous spaces

Let $H \subset G$ be a closed Lie subgroup and consider the homogeneous space G/H. The left-action of G on G/H induces an action of G on the jet space $J^k(\mathbb{R}, G/H)$, called the kth *prolongation* of the action of G on G/H.

Definition 2.4. A *differential relation* (*in one independent variable*) *of order k* on G/H is a submanifold S of $J^k(\mathbb{R}, G/H)$ such that $\mathrm{d}t|_S$ is nowhere vanishing. We define the Pfaffian differential system $(\mathcal{I}, \mathrm{d}t)$ on S, given by restriction to S of the canonical contact system on $J^k(\mathbb{R}, G/H)$. A smooth curve $\gamma : (a, b) \to G/H$ is said to be of *type S* if $j^k(\gamma)|_t \in S$, for all $t \in (a, b)$.

Note that the integral curves of the Pfaffian differential system $(\mathcal{I}, \mathrm{d}t)$ are the k-order jets $j^k(\gamma)$ of curves $\gamma : (a, b) \to G/H$ that satisfy the differential relation $j^k(\gamma)|_t \in S$, for all $t \in (a, b)$.

Definition 2.5. A *Frenet system* of order k on G/H is a triple $(S, \mathbb{A}, \Phi)$, where:

(a) $S \subset J^k(\mathbb{R}, G/H)$ is a G-invariant differential relation of order k endowed with the induced contact system $(\mathcal{I}, \mathrm{d}t)$,
(b) $\mathbb{A} \in P^h(\mathfrak{g})$,
(c) $\Phi : S \to M$ is a smooth equivariant map from S onto an open subset $\Phi(S)$ of $M = G \times \mathbb{A}$, the configuration space,

with the properties that:

- If $\gamma : (a, b) \to G/H$ is a smooth curve of type S then $\Gamma = \Phi \circ j^k(\gamma)$ is an integral curve of the control system $(\mathcal{A}, \omega)$.
- If $\Gamma : (a, b) \to M$ is an integral curve of $(\mathcal{A}, \omega)$ such that $\mathrm{Im}(\Gamma) \subset \Phi(S)$, then $\gamma = \pi_{G/H} \circ \Gamma : (a, b) \to G/H$ is a curve of type S and $\Gamma = \Phi \circ j^k(\gamma)$.

The method of moving frames [11,12] gives an algorithmic procedure for the construction of the Frenet systems for curves of constant type in homogeneous spaces (see [10,17,18]). We refer the reader to [12] for the explicit construction of the Frenet system of generic curves in the affine space $\mathbb{R}^3$, to [7,23] for the Frenet system of generic curves in $\mathbb{RP}^2$, to [20,26] for the Frenet system of generic curves in the conformal 3-sphere and to [22] for the Frenet systems of generic Legendrian curves in the strongly pseudo-convex real hyperquadric Q^3 of $\mathbb{CP}^2$.

Definition 2.6. Let $F := \pi_G \circ \Phi : S \to G$ and $K := \pi_{\mathbb{A}} \circ \Phi : S \to \mathbb{A}$ denote the two components of the map Φ. We call F the *Frenet map* and K the *curvature map*.

Let $\gamma : (a, b) \to G/H$ be a curve of type S, then $\Gamma = \Phi \circ j^k(\gamma) : (a, b) \to M$ is called the *canonical lift* of γ. The maps

$$F_\gamma := F \circ j^k(\gamma) : (a, b) \to G, \qquad K_\gamma := K \circ j^k(\gamma) : (a, b) \to \mathbb{A}$$

are called the *Frenet frame field* and the *curvature function* of γ, respectively.

Definition 2.7. The *generalized arc-length* of a curve $\gamma : (a, b) \to G/H$ of type S is the smooth function $s_\gamma : (a, b) \to \mathbb{R}$, unique up to a constant, such that $\mathrm{d}s_\gamma = \Gamma^*(\omega)$, where $\Gamma : (a, b) \to M$ is the canonical lift of γ. Each curve $\gamma \subset G/H$ of type S may be parameterized in such a way that $\mathrm{d}s_\gamma = \mathrm{d}t$. In this case, we say that the curve γ is *normalized*.

Proposition 2.8. *Let $(S, \mathbb{A}, \Phi)$ be a Frenet system. Then $\Phi(S) = G \times U_\Phi$, where U_Φ is an open subset of $\mathbb{A}$.*

Proof. We set $U_\Phi = \pi_{\mathbb{A}}(\Phi(S))$. Thus, U_Φ is an open subset of $\mathbb{A}$ such that $\Phi(S) \subseteq G \times U_\Phi$. Take any $(g_0, Q_0) \in G \times U_\Phi$. Since $Q_0 \in U_\Phi$ then there exists $g_1 \in G$ such that $(g_1, Q_0) \in \Phi(S)$. Now let $\Gamma : (-\epsilon, \epsilon) \to M$ be an integral curve of the control system $(\mathcal{A}, \omega)$ such that $\Gamma(0) = (g_1, Q_0)$. Since $\Phi(S)$ is an open set we may, by restricting the value of ϵ if necessary, assume that $\mathrm{Im}(\Gamma) \subset \Phi(S)$. Then, the projection of Γ onto G/H is a curve $\gamma : (-\epsilon, \epsilon) \to G/H$ of type S such that $\Gamma = \Phi[j^k(\gamma)]$. Using the G-invariance of S it follows that $g_0 g_1^{-1} \gamma$ is another curve of type S. Thus, from the equivariance of Φ it follows that $g_0 g_1^{-1} \Gamma(0) = (g_0, Q_0)$ belongs to $\Phi(S)$. This shows that $G \times U_\Phi \subseteq \Phi(S)$. □

The elements of the open subset U_Φ may therefore be considered as the "geometrical inputs" of the control system $(\mathcal{A}, \omega)$. In particular, the curvature function K gives a complete set of local differential invariants for curves of type S. More precisely, if $\gamma, \tilde{\gamma} : (a, b) \to G/H$ are normalized curves of type S with $K_\gamma = K_{\tilde{\gamma}}$, then γ and $\tilde{\gamma}$ are congruent to one other, in the sense that there exists a $g \in G$ such that $g\gamma(t) = \tilde{\gamma}(t)$, for all $t \in (a, b)$. Moreover, given any smooth map $K : (a, b) \to U_\Phi \subset \mathbb{A}$ there exists a normalized curve $\gamma : (a, b) \to G/H$ of type S, unique up to congruence, such that $K_\gamma = K$.

If we fix an affine frame $(P, e_1, \ldots, e_h)$ of $\mathbb{A}$ and if we let $k^1, \ldots, k^h$ be the corresponding coordinates, we may identify the configuration space M with $G \times \mathbb{R}^h$. Thus, we may write $K_\gamma = (k^1_\gamma, \ldots, k^h_\gamma)$, where $k^1_\gamma, \ldots, k^h_\gamma$ are smooth functions that depend on the k-jet of γ. These functions can be viewed as the *generalized curvatures* of γ.

2.3. Isotropic curves in $\mathbb{R}^{(2,1)}$

An example that will illustrate our considerations concerns variational principles for isotropic curves in three-dimensional Minkowski space. Let $\mathbb{R}^{(2,1)}$ denote Minkowski 3-space endowed with the Lorentzian inner product

$$\langle v, w\rangle = -(v^1 w^3 + v^3 w^1) + v^2 w^2 =: g_{ij} v^i w^j.$$

We fix the spatial orientation by requiring that the standard basis (e_1, e_2, e_3) is positively oriented, and we fix the time orientation defined by the positive light cone

$$\mathcal{L}^+ = \{v \in \mathbb{R}^{(2,1)} : \langle v, e_1 + e_3\rangle < 0\}.$$

Let G be the restricted Poincaré group $\mathbb{E}(2, 1)$, i.e. the group of isometries of $\mathbb{R}^{(2,1)}$ that preserve the given orientations. The group G may conveniently be described as the space of pairs $g = (q, A)$ where $q \in \mathbb{R}^{(2,1)}$ and $A = (A_1, A_2, A_3)$ is a 3×3 matrix such that

$$\det(A_1, A_2, A_3) = 1, \qquad \langle A_i, A_j\rangle = g_{ij}, \qquad A_1, A_3 \in \mathcal{L}^+.$$

We let $\mathfrak{g}$ denote the Lie algebra of G, consisting of all matrices of the form

$$X(q, v) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^1 & v_1^1 & v_2^1 & 0 \\ q^2 & v_1^2 & 0 & v_2^1 \\ q^3 & 0 & v_1^2 & -v_1^1 \end{pmatrix}.$$

We now define the Maurer–Cartan form $\Omega \in \Omega^1(G, \mathfrak{g})$, which takes the form

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & \omega_1^1 & \omega_2^1 & 0 \\ \omega^2 & \omega_1^2 & 0 & \omega_2^1 \\ \omega^3 & 0 & \omega_1^2 & -\omega_1^1 \end{pmatrix},$$

such that

$$\mathrm{d}q = \omega^i A_i, \qquad \mathrm{d}A_i = \omega_i^j A_j, \quad i = 1, 2, 3.$$

Differentiating these relations, we obtain the structure equations

$$\mathrm{d}\omega^i = -\omega_j^i \wedge \omega^j, \qquad \mathrm{d}\omega_k^i = -\omega_j^i \wedge \omega_k^j, \quad i, k = 1, 2, 3.$$

Recall that the Maurer–Cartan forms $\omega^1, \omega^2, \omega^3, \omega_1^1, \omega_1^2, \omega_2^1$ are linearly independent and generate the space, $\mathfrak{g}^*$, of left-invariant 1-forms on G.

Definition 2.9. A *null* (*or isotropic*) *curve* in $\mathbb{R}^{(2,1)}$ is a smooth parameterized curve

$$\alpha : (a, b) \subset \mathbb{R} \to \mathbb{R}^{(2,1)}$$

such that $\alpha'(t) \in \mathcal{L}^+$ for all $t \in (a, b)$. We shall assume that α is without flex points, in the sense that

$$\alpha'(t) \wedge \alpha''(t) \neq 0 \quad \forall t \in (a, b).$$

The linear differential form $\omega_\alpha := \|\alpha''(t)\|^{1/2}\,\mathrm{d}t$ is nowhere vanishing, and is invariant under changes of parameter and the action of the group G. Without loss of generality we may assume that α is *normalized*, in the sense that

$$\|\alpha''(t)\|^{1/2} = 1 \quad \forall t \in (a, b).$$

(This condition fixes the parameter t up to an additive constant.) The *curvature* of α is defined by

$$k(t) = -\tfrac{1}{2}\|\alpha'''(t)\|^2 \quad \forall t \in (a, b).$$

At each point of the curve we may define the frame $g(t) = (\alpha(t), A(t)) \in G$ given by

$$A_1(t) = \alpha'(t), \qquad A_2(t) = \alpha''(t), \qquad A_3(t) = \alpha'''(t) + \tfrac{1}{2}\|\alpha'''(t)\|^2\alpha'(t).$$

This frame defines a canonical lift

$$g : t \in (a, b) \to g(t) = (\alpha(t), A(t)) \in G$$

of the curve α to the group G, referred to as the *Frenet frame field* along α. An application of the method of moving frames shows that the Frenet frame field is the unique lift of α to G with the property that

$$g^*(\Omega) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & \kappa & 0 \\ 0 & 1 & 0 & \kappa \\ 0 & 0 & 1 & 0 \end{pmatrix} \mathrm{d}t.$$

We illustrate the construction of the Frenet system for isotropic curves in $\mathbb{R}^{(2,1)}$, viewed as a homogeneous space of the group G. Let

$$t, \quad X = (x^1, x^2, x^3), \quad X_1 = (x_1^1, x_1^2, x_1^3), \quad X_2 = (x_2^1, x_2^2, x_2^3), \quad X_3 = (x_3^1, x_3^2, x_3^3)$$

be the standard coordinates on the jet space $J^3(\mathbb{R}, \mathbb{R}^{(2,1)}) \cong \mathbb{R} \times \mathbb{R}^{(2,1)} \times \mathbb{R}^{(2,1)} \times \mathbb{R}^{(2,1)} \times \mathbb{R}^{(2,1)}$. The differential relation $S \subset J^3(\mathbb{R}, \mathbb{R}^{(2,1)})$ is defined by

$$X_1 \in \mathcal{L}^+, \qquad \|X_2\| = 1, \qquad (X_1, X_2) = (X_2, X_3) = 0, \qquad X_1 \wedge X_2 \wedge X_3 \neq 0.$$

Holonomic sections of S are third-order jets $j^3(\alpha)$ of normalized isotropic curves $\alpha : (a, b) \to \mathbb{R}^{(2,1)}$. We define $\kappa : S \to \mathbb{R}$ by

$$\kappa(t, X, X_1, X_2, X_3) = -\tfrac{1}{2}\|X_3\|^2.$$

$\kappa[j^3(\alpha)]$ is then the curvature of the isotropic curve α.

The affine space $\mathbb{A} = P + \mathfrak{a} \subset \mathfrak{g}$ is then the straight line

$$k \in \mathbb{R} \to Q(k) = e_0 + ke_1 \in \mathfrak{g},$$

where

$$e_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is convenient to identify the configuration space $M = G \times \mathbb{A}$ with $G \times \mathbb{R}$ by means of the map

$$(g, e_0 + ke_1) \in G \times \mathbb{A} \to (g, k) \in G \times \mathbb{R}.$$

With this identification at hand, the linear control system $(\mathcal{A}, \omega)$ is generated by the linear differential forms

$$\eta^1 = \omega^1_2 - k\omega, \qquad \eta^2 = \omega^1_1, \qquad \eta^3 = \omega^2_1 - \omega, \qquad \eta^4 = \omega^2, \qquad \eta^5 = \omega^3,$$

along with independence condition

$$\omega = \omega^1.$$

We now consider a smooth curve $\Gamma : (a, b) \to M$ and let

$$g : t \in (a, b) \to (\alpha(t), A(t)) \in G, \qquad k : t \in (a, b) \to k(t) \in \mathbb{A},$$

be the two components of Γ. Then Γ is an integral curve of the control system $(\mathcal{A}, \omega)$ if and only if $g : (a, b) \to G$ is the Frenet field along the isotropic curve $\alpha : (a, b) \to \mathbb{R}^{(2,1)}$, and k is the curvature of the curve α. The mapping $\Phi : S \to M = G \times \mathbb{R}$ linearizing the differential relation S is defined by

$$S : (t, X, X_1, X_2, X_3) \in S \to ((X; X_1, X_2, X_3 + \tfrac{1}{2}\|X_3\|^2 X_1), -\tfrac{1}{2}\|X_3\|^2) \in M.$$

Remark 2.10. Using the structure equations for $\mathbb{E}(2, 1)$ we find that

$$\mathrm{d}\omega = (\kappa\eta^4 - \eta^2) \wedge \omega - \eta^1 \wedge \eta^4, \tag{1a}$$

$$\mathrm{d}\eta^1 = -\pi \wedge \omega + \eta^1 \wedge \eta^2 + \kappa\eta^1 \wedge \eta^4, \tag{1b}$$

$$\mathrm{d}\eta^2 = (\kappa\eta^3 - \eta^1) \wedge \omega - \eta^1 \wedge \eta^3, \tag{1c}$$

$$\mathrm{d}\eta^3 = (2\eta^2 - \kappa\eta^4) \wedge \omega + \eta^1 \wedge \eta^4 + \eta^2 \wedge \eta^3, \tag{1d}$$

$$\mathrm{d}\eta^4 = (\kappa\eta^5 - \eta^4) \wedge \omega - \eta^1 \wedge \eta^5, \tag{1e}$$

$$\mathrm{d}\eta^5 = \eta^4 \wedge \omega + \eta^2 \wedge \eta^5 - \eta^3 \wedge \eta^4, \tag{1f}$$

where

$$\pi = \mathrm{d}\kappa + \kappa^2\eta^4.$$

2.4. Coadjoint action of $\mathbb{E}(2,1)$

For later convenience, we now discuss the coadjoint action of $\mathbb{E}(2,1)$ on $\mathfrak{e}(2,1)^*$, the dual of its Lie algebra. Our discussion follows the discussion of the coadjoint representation of $\mathbb{E}(3)$ given in Guillemin and Sternberg [14].

Using the Maurer–Cartan forms, we identify $\mathfrak{g}^*$ with $\mathbb{R}^{(2,1)} \oplus \mathbb{R}^{(2,1)}$ by means of the map

$$(p, v) \in \mathbb{R}^{(2,1)} \oplus \mathbb{R}^{(2,1)} \to p_i\omega^i - v^1\omega_1^2 + v_2\omega_1^1 + v_3\omega_2^1 \in \mathfrak{g}^*.$$

The coadjoint action of G on $\mathfrak{g}^*$ then takes the form

$$g \cdot (p, v) = (Ap, Av - (Ap) \times q), \tag{2}$$

for all

$$g = \begin{pmatrix} 1 & 0 \\ q & A \end{pmatrix} \in G = \mathbb{E}(2,1),$$

where $\times$ denotes the vector cross product

$$\langle v \times w, u\rangle = \det(v, w, u) \quad \forall v, w, u \in \mathbb{R}^{(2,1)}.$$

We now define the map

$$C : (p, v) \in \mathfrak{g}^* \to (\|p\|^2, \langle p, v\rangle) \in \mathbb{R}^2,$$

the components which, C_1 and C_2, generate the space of Casimir functions. We recall the following standard material:

- Let G be a Lie group, and $\mathfrak{g}^*$ the dual of the Lie algebra of G. Let $\mu \in \mathfrak{g}^*$. The *isotropy group* of G at μ is the closed subgroup of G defined by

$$G_\mu := \{g \in G : \mathrm{Ad}^*(g)\mu = \mu\} = \{g \in G : \langle \mu; \mathrm{Ad}(g^{-1})A\rangle = \langle \mu; A\rangle,\ \forall A \in \mathfrak{g}\}.$$

- The Lie algebra of G_μ is

$$\mathfrak{g}_\mu = \{A \in \mathfrak{g} : \mathrm{ad}^*(A)\mu = 0\} = \{A \in \mathfrak{g} : \langle \mu; [A, B]\rangle = 0,\ \forall B \in \mathfrak{g}\}.$$

- The *rank* of the group G is defined as

$$\mathrm{rank}(G) = \inf\{\dim(\mathfrak{g}_\mu) : \mu \in \mathfrak{g}^*\}.$$

- An element $\mu \in \mathfrak{g}^*$ is *regular* if $\dim(\mathfrak{g}_\mu) = \mathrm{rank}(G)$, otherwise μ is a *singular* element of $\mathfrak{g}^*$. The set of regular elements of $\mathfrak{g}^*$ will be denoted by $\mathfrak{g}^*_\mathrm{r}$, while $\mathfrak{g}^*_\mathrm{s}$ will denote the set of singular elements.
- By a theorem of Dixmier (cf. [8,9]), the isotropy group G_μ and the isotropy Lie algebra $\mathfrak{g}_\mu$ of a regular element $\mu \in \mathfrak{g}^*_\mathrm{r}$ are Abelian.

  In the case of $\mathbb{E}(2,1)$, $\mathfrak{g}^*_\mathrm{r}$ is the open subset of $\mathfrak{g}^*$ consisting of elements

$$\mathfrak{g}^*_\mathrm{r} = \{(p, v) \in \mathfrak{g} : p \neq 0\}.$$

The co-adjoint orbit, $\mathcal{O}_{(p_0,v_0)}$, through a regular element $(p_0, v_0) \in \mathfrak{g}_{\mathrm{r}}^*$ is therefore the four-dimensional sub-manifold

$$\mathcal{O}_{(p_0,v_0)} = \{(p, v) \in \mathbb{R}^{(2,1)} \oplus \mathbb{R}^{(2,1)} : \|p\|^2 = \|p_0\|^2, \langle p_0, v_0\rangle = \langle p, v\rangle\}.$$

There are three types of regular orbit:

- orbits of positive type: $\mathcal{O}_{(p_0,v_0)}$ with $C_1 = \|p_0\|^2 > 0$;
- orbits of negative type: $\mathcal{O}_{(p_0,v_0)}$ with $C_1 = \|p_0\|^2 < 0$;
- orbits of null type: $\mathcal{O}_{(p_0,v_0)}$ with $C_1 = \|p_0\|^2 = 0$;

The orbits of negative and null type also break into sub-classes according to whether p_0 is future-directed, with $\langle p_0, e_1 + e_3\rangle < 0$, or past-directed, with $\langle p_0, e_1 + e_3\rangle > 0$.

3. Variational problems

3.1. Non-degenerate invariant variational problems

Definition 3.1. Given an affine subspace $\mathbb{A} \in P^h(\mathfrak{g})$, an *invariant Lagrangian of type* $\mathbb{A}$ is a smooth real-valued function $L : \mathbb{A} \to \mathbb{R}$.

An invariant Lagrangian L gives rise to a variational problem defined on the integral curves of the linear control system $(\mathcal{A}, \omega)$. From this point of view the Lagrangian L is considered as a *cost function*. It is then an optimal control problem to minimize the cost

$$\mathcal{L} : \Gamma \to \int_\gamma \Gamma^*(L\omega)$$

among the integral curves of $(\mathcal{A}, \omega)$. If $(\mathcal{A}, \omega)$ comes from a Frenet system $(S, \mathbb{A}, \Phi)$ on the homogeneous space G/H then the Lagrangian L defines a geometric action functional $\tilde{\mathcal{L}} : \mathcal{S} \to \mathbb{R}$ acting on the space of the normalized curves of type S:

$$\tilde{\mathcal{L}} : \gamma \in \mathcal{S} \to \int_\gamma L(K[j^k(\gamma)(t)])\,\mathrm{d}t.$$

Note that the geometric action functional $\tilde{\mathcal{L}}$ depends only on the generalized curvatures of γ.

Example 3.2. The simplest invariant variational problem for a Frenet systems is the *arc-length functional*, which is defined by a constant Lagrangian (see [7,12,20,22,23] for more details about the arc-length functionals for generic curves in the conformal and pseudo-conformal three-dimensional sphere, in the real projective plane and in the affine plane). Another typical example of an invariant Lagrangian is the *Kirchhoff variational problem* for the Frenet system of generic curves in $\mathbb{R}^3$, defined by the action functional

$$\mathcal{L} : \gamma \subset \mathbb{R}^3 \to \int_\gamma (\kappa^2(u) + a\tau(u))\,\mathrm{d}u.$$

The extremal curves are the canonical lifts of the Kirchhoff elastic rods of $\mathbb{R}^3$. When $a = 0$, we get the total squared curvature functional. Other examples of invariant variational problems for curves in $\mathbb{R}^3$ have been considered in Ref. [19].

Given an invariant Lagrangian $L : \mathbb{A} \to \mathbb{R}$, we construct the corresponding affine sub-bundle $\tilde{Z} \subset T^*(M)$ over the configuration space $M = G \times \mathbb{A}$. The fiber of $\tilde{Z}$ over the point $(g, Q) \in M$ is given by the affine space

$$\tilde{Z}|_{(g,Q)} = \{\eta \in \mathfrak{g}^* : \langle \eta; Q\rangle = L(Q)\}.$$

Note that $\tilde{Z}$ is of the form $G \times \tilde{\mathcal{Z}}$, where

$$\tilde{\mathcal{Z}} = \{(Q, \eta) \in \mathbb{A} \times \mathfrak{g}^* : \langle \eta; Q\rangle = L(Q)\}.$$

The Liouville 1-form ψ is given by

$$\psi|_{(g,Q,\eta)} = \pi^*(\eta)|_{(g,Q,\eta)} \quad \forall (g, Q, \eta) \in \tilde{Z},$$

where $\pi : G \times \tilde{\mathcal{Z}} \to G$ denotes the projection onto the first factor.

Remark 3.3. Pick a basis $(e_0, e_1, \ldots, e_h, e_{h+1}, \ldots, e_n)$ of $\mathfrak{g}$ such that

$$P = e_0, \qquad \mathfrak{a} = \mathrm{span}(e_1, \ldots, e_h), \qquad \omega = \theta^0,$$

where $(\theta^0, \ldots, \theta^n)$ is the dual basis of $\mathfrak{g}^*$. We use the following index range: $i, j = 1, \ldots, h$, $a, b = h + 1, \ldots, n$. The map

$$\begin{aligned}(g, k, \lambda) \in G \times \mathbb{R}^h \times \mathbb{R}^n \to (g, e_0 + k^j e_j, L(e_0 + k^j e_j)\omega \\ + \lambda_j(\theta^j - k^j\omega) + \lambda_a\theta^a) \in \tilde{Z}\end{aligned}$$

gives an explicit identification between $G \times \mathbb{R}^h \times \mathbb{R}^n$ and $\tilde{Z}$. With this identification at hand, the tautological 1-form can be written as

$$\psi = (L(k^1, \ldots, k^h) - k^j\lambda_j)\omega + \lambda_j\theta^j + \lambda_a\theta^a.$$

Definition 3.4. An invariant Lagrangian $L : \mathbb{A} \to \mathbb{R}$ is said to be *regular* if the corresponding variational problem $(\mathcal{A}, \omega, L)$ is regular, i.e. if the Cartan system of $\Psi = \mathrm{d}\psi$, with the independence condition ω, is reducible (see Definition A.6). For a regular Lagrangian we denote by $Y \subset \tilde{Z}$ the momentum space of the variational problem $(\mathcal{A}, \omega, L)$.

Remark 3.5. We have seen that all the derived systems of $(\mathcal{A}, \omega)$ have constant rank. This implies that the extremal curves of a regular invariant variational problem are the projections of the integral curves of the Euler–Lagrange system on Y (cf. [3]).

Proposition 3.6. *Let $L : \mathbb{A} \to \mathbb{R}$ be a regular Lagrangian with momentum space Y. Then $Y = G \times \mathcal{F}$, where $\mathcal{F}$ is an immersed submanifold of $\mathbb{A} \times \mathfrak{g}^*$.*

Proof. First we claim that the momentum space, Y, is G-invariant. To show this, for any $g \in G$, we consider the submanifold $g \cdot Y \subset \tilde{Z}$. The G-invariance of the exterior differential forms ψ, Ψ and ω implies that left translation $L_g : \tilde{Z} \to \tilde{Z}$ sends integral elements of $(\mathcal{C}(\Psi), \omega)$ into integral elements of $(\mathcal{C}(\Psi), \omega)$. Hence, for every point $p \in g \cdot Y$, there exists an integral element of $(\mathcal{C}(\Psi), \omega)$ tangent to $g \cdot Y$. Since the momentum space Y is maximal with respect to this property, it follows that $g \cdot Y \subseteq Y$. Thus the group G acts on Y. Since this action is free and proper, the quotient space $\mathcal{F} = Y/G$ exists as a manifold. The natural projection $\pi : Y \to \mathcal{F}$ is constant along the fibers of the map $(g, Q, \eta) \in Y \to (Q, \eta) \in \mathbb{A} \times \mathfrak{g}^*$. Thus it induces a smooth one-to-one immersion $j : \mathcal{F} \to \mathbb{A} \times \mathfrak{g}^*$. We conclude the proof by observing that the map $(\mathrm{id}, j) : G \times \mathcal{F} \to Y$ is a smooth diffeomorphism. □

Definition 3.7. We call $\mathcal{F}$ the *phase space* of the system. Note that a point $p \in \mathcal{F}$ is of the form $p = (Q, \eta)$, where $Q \in \mathbb{A}$, $\eta \in \mathfrak{g}^*$. We define the maps

$$\Lambda : (Q, \eta) \in \mathcal{F} \to \eta \in \mathfrak{g}^*, \qquad \mathcal{H} : (Q, \eta) \in \mathcal{F} \to Q \in \mathbb{A} \subset \mathfrak{g}.$$

We refer to Λ as the *Legendre transform* and $\mathcal{H}$ as the *Hamiltonian*. Let $F(p) := T_p(\mathcal{F}) \subset \mathfrak{a} \oplus \mathfrak{g}^*$ be the tangent space of $\mathcal{F}$ at p. We then define

$$R(p) := \mathrm{d}\Lambda|_p[F(p)] \subset \mathfrak{g}^*, \qquad S(p) := \mathrm{d}\mathcal{H}|_p[F(p)] \subset \mathfrak{a} \quad \forall p \in \mathcal{F}.$$

Definition 3.8. A regular invariant Lagrangian $L : \mathbb{A} \to \mathbb{R}$ is said to be *non-degenerate* if the momentum space Y is odd-dimensional, of dimension $2m + 1$, and if the restriction of the canonical 2-form Ψ to Y, Ψ_Y, has the property that $\omega \wedge (\Psi_Y)^m$ is non-vanishing.

Examples of invariant non-degenerate variational problems include the total squared curvature functional in two- and three-dimensional space forms [5,12], the Kirchhoff variational problem in $\mathbb{R}^3$, the Poincareé and the Delaunay functionals [12,16,21], the projective, the conformal and the pseudo-conformal arc-length functionals (cf. [7,20,22]).

Given a non-degenerate variational problem, it follows that $\omega \wedge (\Psi_Y)^m$ defines a volume form on Y, and that Ψ_Y is of maximal rank on Y. Therefore there exists a unique vector field $\xi \in \mathfrak{X}(Y)$ such that $i_\xi(\Psi_Y) = 0$ and $\omega(\xi) = 1$.

Definition 3.9. ξ is the *characteristic vector field* of the non-degenerate variational problem $(\mathcal{A}, \omega, L)$.

If $(\mathcal{A}, \omega, L)$ is non-degenerate then the Euler–Lagrange system is simply the Cartan system of the canonical 2-form restricted to the momentum space: $\mathcal{E} = \mathcal{C}(\Psi_Y)$. Therefore, for such variational problems, the integral curves of the Euler–Lagrange system are the integral curves of the characteristic vector field ξ (see Ref. [12]). We therefore have the following theorem.

Theorem 3.10. *Let $\Gamma : (a, b) \to Y$ be an integral curve of the characteristic vector field ξ of a non-degenerate variational problem $(\mathcal{A}, \omega, L)$. Then $\gamma = \pi_M \circ \Gamma : (a, b) \to M$ is a critical point of the action functional $\mathcal{L}$.*

Proposition 3.11. *If L is non-degenerate then the Legendre transform $\Lambda : \mathcal{F} \to \mathfrak{g}^*$ is an immersion.*

Proof. Let ξ be the characteristic vector field of the momentum space. The Liouville form ψ, the canonical 2-form Ψ and the independence condition ω are G-invariants, therefore the characteristic vector field is also G-invariant. Since $\xi|_{(g,p)} \in T_{(g,p)}Y \cong \mathfrak{g} \oplus F(p) \subset \mathfrak{g} \oplus \mathfrak{a} \oplus \mathfrak{g}^*$, this implies that there exist smooth maps $A_\xi : \mathcal{F} \to \mathfrak{g}$ and $\Phi_\xi : \mathcal{F} \to \mathfrak{a} \oplus \mathfrak{g}^*$ with the property that

$$\xi|_{(g,p)} = A_\xi(p)|_g + \Phi_\xi(p) \quad \forall (g, p) \in Y,$$

where $\Phi_\xi(p) \in F(p)$ for all $p \in \mathcal{F}$. Since ξ satisfies the transversality condition $1 = \omega(\xi) = \langle \omega; A_\xi \rangle$, then $A_\xi : \mathcal{F} \to \mathfrak{g}$ is a nowhere vanishing function. If we now consider $\{0\} \oplus \ker[\mathrm{d}\Lambda|_p] \subset \mathfrak{g} \oplus F(p)$ then it is simple to check that every such vector lies in the kernel of the canonical 2-form Ψ. Since this null-distribution is generated by ξ, we therefore have

$$\{0\} \oplus \ker[\mathrm{d}\Lambda|_p] \subset \mathrm{span}[A_\xi(p) + \Phi_\xi(g)].$$

Since A_ξ is non-vanishing, however, this holds if and only if $\ker[\mathrm{d}\Lambda|_p] = \{0\}$. □

From now on we will assume that the Legendre transform Λ is a one-to-one immersion, so that the phase space, $\mathcal{F}$, can be considered as a submanifold (not necessarily embedded) of $\mathfrak{g}^*$. Consequently, we will think of the momentum space as an immersed submanifold of $G \times \mathfrak{g}^*$. The notation introduced in the preceding paragraphs can then be simplified as follows:

- the Legendre map Λ is the inclusion of $\mathcal{F}$ into $\mathfrak{g}^*$;
- the tangent space $F(\eta)$ of $\mathcal{F}$ at $\eta \in \mathcal{F}$ is a linear subspace of $\mathfrak{g}^*$ and $R(\eta) = F(\eta)$;
- the tangent space $T_{(g,\eta)}(Y)$ is identified with $\mathfrak{g} \oplus F(\eta) \subset \mathfrak{g} \oplus \mathfrak{g}^*$;
- the Liouville form and the canonical 2-form on Y are the restrictions to Y of the Liouville form and the standard symplectic form on $T^*(G)$;
- the characteristic vector field ξ can be written as

  $$\xi|_{(g,\eta)} = A_\xi(\eta)|_g + \Phi_\xi(\eta) \quad \forall (g, \eta) \in Y,$$

  where $A_\xi : \mathcal{F} \to \mathfrak{g}$ and $\Phi_\xi : \mathcal{F} \to \mathfrak{g}^*$ are smooth functions such that $\Phi_\xi(\eta) \in F(\eta)$, for all $\eta \in \mathcal{F}$.

From now on, we will adhere to these simplifications.

With this notation at hand, we may use the left-invariant trivialization of $T(G)$ to identify the tangent space

$$T_{(g,\eta)}(Y) \cong T_gG \oplus T_\eta\mathcal{F} \cong \mathfrak{g} \oplus F(\eta) \subset \mathfrak{g} \oplus \mathfrak{g}^*.$$

We then have the explicit isomorphism

$$A + v \in \mathfrak{g} \oplus F(\eta) \to A|_g + v \in T_{(g,\eta)}(Y),$$

where $A \in \mathfrak{g} = T_{id}(G)$ and $A|_g = (L_g)_*A \in T_g(G)$. With this identification, the Liouville form ψ becomes the cross-section of $T^*(Y)$ defined by

$$\psi|_{(g,\eta)}(A + v) = \langle \eta, A \rangle \quad \forall (g, \eta) \in Y \;\; \forall A + v \in \mathfrak{g} \oplus F(\eta). \tag{3}$$

Then, from the standard formula

$$\mathrm{d}\psi(X, Y) = \tfrac{1}{2}\{X[\psi(Y)] - Y[\psi(X)] - \psi([X, Y])\},$$

it follows that the canonical 2-form $\Psi = \mathrm{d}\psi \in \Omega^2(Y)$ takes the form

$$\Psi|_{(g,\eta)}(A + v; B + w) = -\tfrac{1}{2}\langle w; A\rangle + \tfrac{1}{2}\langle \mathrm{ad}^*(A)\eta + v; B\rangle, \tag{4}$$

for all $\eta \in \mathcal{F}$ and for all $A + v, B + w \in \mathfrak{g} \oplus F(\eta)$.

Definition 3.12. Given a left-invariant 1-form $\mu \in \mathfrak{g}^*$, let $\mathcal{O}(\mu) \subset \mathfrak{g}^*$ be the coadjoint orbit passing through μ, and let $\mathrm{O}(\mu) := \mathrm{ad}^*(\mathfrak{g})\mu \subset \mathfrak{g}^*$ denote the tangent space to the orbit $\mathcal{O}(\mu)$ at μ. The *linearized phase portrait* of the point $\eta \in \mathcal{F}$ is the linear subspace $\Pi(\eta) := F(\eta) \cap \mathrm{O}(\eta)$ of $\mathfrak{g}^*$. The subset $\mathcal{P}(\mu) = \mathcal{F} \cap \mathcal{O}(\mu)$ is referred to as the *phase portrait* of $\mu \in \mathfrak{g}^*$.

The following result shows that the characteristic vector field ξ may be written in terms of the Hamiltonian $\mathcal{H}$:

Theorem 3.13. *The characteristic vector field ξ is given by*

$$\xi|_{(g,\eta)} = \mathcal{H}(\eta)|_g - \mathrm{ad}^*[\mathcal{H}(\eta)]\eta \quad \forall (g, \eta) \in Y. \tag{5}$$

Proof. Given a point $\eta \in \mathcal{F}$, we set

$$\mathrm{Ann}(F(\eta)) = \{A \in \mathfrak{g} : \langle v; A\rangle = 0,\ \forall v \in F(\eta)\}$$

and let

$$\rho(\eta) : \mathrm{Ann}(F(\eta)) \to \mathrm{O}(\eta)$$

be the linear map

$$\rho(\eta) : A \in \mathrm{Ann}(F(\eta)) \to \mathrm{ad}^*(A)\eta \in \mathrm{O}(\eta). \tag{6}$$

It then follows from Eqs. (3) and (4) that a tangent vector $A + v \in \mathfrak{g} \oplus F(\eta)$ to the momentum space Y at the point (g, η) belongs to the kernel of Ψ if and only if

$$A \in \rho(\eta)^{-1}(\Pi(\eta)), \qquad v = -\rho(\eta)A. \tag{7}$$

We now let $(g_0, \eta_0) \in Y$ and let $\Gamma : (-\epsilon, \epsilon) \to Y$ be the integral curve of the characteristic vector field ξ with initial condition $\Gamma(0) = (g_0, \eta_0)$. We write $\Gamma(t) = (g(t), \eta(t))$, where $g : (-\epsilon, \epsilon) \to G$ and $\eta : (-\epsilon, \epsilon) \to \mathcal{F}$ are smooth maps such that

$$g(t)^{-1}g'(t)\,\mathrm{d}t = g^*(\Theta)|_t, \quad g^{-1}(t)g'(t) = A_\xi[\eta(t)], \quad g(0) = g_0, \quad \eta(0) = \eta_0.$$

On the other hand[3]

$$t \in (-\epsilon, \epsilon) \to (g(t), \mathcal{H}[\eta(t)]) \in G \times \mathbb{A} = M$$

[3] It is a general fact that if $\pi : Y \to M$ is the momentum space of a regular variational problem $(\mathcal{I}, \omega, L)$ on the configuration space M and if $\Gamma : (a, b) \to Y$ is an integral curve of the Euler–Lagrange system, then $\gamma = \pi \circ \Gamma$ is an integral curve of the Pfaffian differential system $(\mathcal{I}, \omega)$ on M (see [12]).

is an integral curve of the linear control system $(\mathcal{A}, \omega)$. We then have

$$g^*(\Theta)|_t = \mathcal{H}[\eta(t)]g^*(\omega)|_t = \mathcal{H}[\eta(t)]\mathrm{d}t|_t.$$

Therefore, we conclude that

$$A_\xi[\eta(t)] = \mathcal{H}[\eta(t)] \quad \forall t \in (-\epsilon, \epsilon).$$

Since ξ belongs to the kernel of Ψ, we conclude from Eq. (7) that

$$\Phi_\xi[\eta(t)] = -\mathrm{ad}^*[\mathcal{H}(\eta(t))]\eta(t) \quad \forall t \in (-\epsilon, \epsilon).$$

This yields the required result. □

Definition 3.14. The *phase flow* is the flow of the vector field $\Phi_\xi : \eta \in \mathcal{F} \to -\mathrm{ad}^*[\mathcal{H}(\eta)]\eta \in \mathfrak{g}^*$.

Remark 3.15. We use the notation $\phi_\xi : \mathcal{D} \subset \mathbb{R} \times \mathcal{F} \to \mathcal{F}$ to indicate the phase flow. We observe the following facts:

- The domain of definition $\mathcal{D}$ of the phase flow is of the form

  $$\mathcal{D} = \{(t, \eta) \in \mathbb{R} \times \mathcal{F} : t \in (\epsilon^-(\eta), \epsilon^+(\eta))\},$$

  where $\epsilon^- : \mathcal{F} \to \mathbb{R}^- \cup \{-\infty\}$ and $\epsilon^+ : \mathcal{F} \to \mathbb{R}^+ \cup \infty$.
- For every $\eta \in \mathcal{F}$, the curve

  $$\phi_\eta : (\epsilon^-(\eta), \epsilon^+(\eta)) \to \phi_\xi(t, \eta) \in \mathcal{F}$$

  is the maximal integral curve of Φ_ξ with the initial condition $\phi_\eta(0) = \eta$.
- $\Phi_\xi(\eta) \in \Pi(\eta)$ and $\phi_\eta(t) \in \mathcal{P}(\eta)$, for every $\eta \in \mathcal{F}$ and every $t \in (\epsilon^-(\eta), \epsilon^+(\eta))$.
- If we fix a point $(g_0, \eta_0) \in Y = G \times \mathcal{F}$, then the maximal integral curve of the characteristic vector field ξ with the initial condition (g_0, η_0) is given by

  $$\Gamma_{(g_0,\eta_0)} : t \in (\epsilon^-(\eta_0), \epsilon^+(\eta_0)) \to (h_{(g_0,\eta_0)}(t), \phi_{\eta_0}(t)) \in G \times \mathcal{F},$$

  where $h_{(g_0,\eta_0)}$ is the (unique) solution of the equation

  $$h^{-1}h' = \mathcal{H}[\phi_{\eta_0}(t)], \quad h(0) = g_0.$$

- We set $\tilde{\mathcal{D}} = \{(t; (g, \eta)) \in \mathbb{R} \times Y : t \in (\epsilon^-(\eta), \epsilon^+(\eta))\}$. The flow Γ of the characteristic vector field ξ is the local one-parameter group of transformations $\Gamma : \tilde{\mathcal{D}} \subset \mathbb{R} \times Y \to Y$ given by

  $$\Gamma(t, g, \eta) = (h_{(g,\eta)}(t), \phi(t, \eta)) \quad \forall (t; (g, \eta)) \in \tilde{\mathcal{D}}.$$

Remark 3.16. The phase flow $\phi_\xi : \mathcal{D} \to \mathcal{F}$ satisfies the *Euler equation*

$$\left.\frac{\partial \phi_\xi}{\partial t}\right|_{(t,\eta)} = -\mathrm{ad}^*[\mathcal{H}[\phi_\xi(t, \eta)]]\phi_\xi(t, \eta), \quad \phi_\xi(0, \eta) = \eta, \ \forall (t, \eta) \in \mathcal{D}. \tag{8}$$

If there exists a G-equivariant isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$, then we can identify $\mathfrak{g}$ and $\mathfrak{g}^*$. (For example, if G is semisimple, then take the pairing defined by the Killing form of $\mathfrak{g}$.) Using this identification, the Euler equation can be written in the Lax form

$$\left.\frac{\partial \phi_\xi}{\partial t}\right|_{(t,\eta)} = -[\mathcal{H}[\phi_\xi(t,\eta)], \phi_\xi(t,\eta)], \quad \phi_\xi(0,\eta) = \eta, \ \ \forall (t,\eta) \in \mathcal{D}.$$

Definition 3.17. We denote by $\mathcal{F}_{\mathrm{s}} = \{\eta \in \mathcal{F} : \Phi_\xi(\eta) = 0\}$ the set of all fixed points of the phase flow and by $\mathcal{F}_{\mathrm{r}}$ the complement of $\mathcal{F}_{\mathrm{s}}$. We call $\mathcal{F}_{\mathrm{s}}$ and $\mathcal{F}_{\mathrm{r}}$ the *singular* and the *regular* parts of the phase space, respectively. We call $\Sigma = G \times \mathcal{F}_{\mathrm{s}} \subset Y$ the *bifurcation set* and refer to $Y_{\mathrm{r}} = Y \setminus \Sigma$ as the *regular part* of the momentum space. The intersection $\mathcal{P}_{\mathrm{r}}(\mu) = \mathcal{F}_{\mathrm{r}} \cap \mathcal{O}(\mu) \subset \mathcal{P}(\mu)$ is called the *regular part* of the phase portrait $\mathcal{P}(\mu)$. The connected component $\tilde{\mathcal{P}}(\mu)$ of $\mathcal{P}_{\mathrm{r}}(\mu)$ containing μ is referred to as the *reduced phase portrait* of μ.

The following result, which may be verified by applying the uniqueness theorems for ordinary differential equations, characterizes integral curves of the characteristic vector field that intersect the bifurcation set.

Proposition 3.18. *Let $p = (g, \eta) \in \Sigma$ be a point of the bifurcation set. The integral curve $\Gamma_\xi(-, p) : \mathbb{R} \to Y$ of the characteristic vector field ξ passing through p is the orbit of the one-parameter subgroup generated by $\mathcal{H}(\eta)$:*

$$\Gamma_\xi(t, p) = (\exp(\mathcal{H}(\eta)t)g, \eta) \quad \forall t \in \mathbb{R}.$$

This result implies that if $(\mathcal{A}, \omega)$ comes from a Frenet system of curves in G/H, then the curve $\gamma \subset G/H$ of type S that corresponds to $\Gamma_\xi(-, p)$, where $p \in \Sigma$, has constant curvature (i.e. $K_\gamma =$ constant).

Since this result completely characterizes the behavior of integral curves that intersect the bifurcation set Σ, we shall henceforth restrict our attention to the regular parts of the phase space and momentum space. Therefore, to simplify the notation, $\mathcal{F}$, Y and $\mathcal{P}(\mu)$ will be used to denote the regular parts of the phase space, the momentum space and the phase portraits, respectively.

3.2. The Poincaré variational problem for isotropic curves in $\mathbb{R}^{(2,1)}$

We now return to our example of isotropic curves in $\mathbb{R}^{(2,1)}$ considered in Section 2.3. Let m be a non-zero constant and consider the variational problem on the space $\mathcal{V}$ of parameterized integral curves $\Gamma : t \in (a, b) \to (g(t), k(t)) \in G \times \mathbb{R}$ of the Pfaffian system $(\mathcal{A}, \omega)$ defined by the action functional

$$\mathcal{L}_m : \Gamma \in \mathcal{V} \to \int_\Gamma (1 + mk)\omega.$$

Geometrically, this amounts to an analogue of the Poincaré variational problem where we minimize the arc-length functional (defined by the integral of the canonical line-element of the null curve) amongst normalized null curves $\alpha \subset \mathbb{R}^{(2,1)}$ subject to the additional constraint that the integral of the curvature k along the curve be held constant.

The affine sub-bundle $\tilde{Z} \subset T^*(M)$ is given by $M \times \tilde{\mathcal{Z}}$, where $\tilde{\mathcal{Z}} \subset \mathfrak{g} \oplus \mathfrak{g}^*$ is the submanifold consisting of all $(Q(k), \eta) \in \mathbb{A} \oplus \mathfrak{g}^*$ such that $\langle \eta; Q(k) \rangle = 1 + mk$. (See Section 2.3 for the definition of the map $Q : \mathbb{R} \to \mathfrak{g}$.) Therefore $(Q(k), \eta)$ belongs to $\tilde{\mathcal{Z}}$ if and only if

$$\eta = \eta(k, \lambda_1, \dots, \lambda_5) := (1 + mk)\omega + \lambda_1\eta^1 + \lambda_2\eta^2 + \lambda_3\eta^3 + \lambda_4\eta^4 + \lambda_5\eta^5,$$

where $\lambda_1, \dots, \lambda_5 \in \mathbb{R}$. For simplicity, we identify $\tilde{Z}$ with $G \times \mathbb{R}^6$ by means of the map

$$(g : k, \lambda_1, \dots, \lambda_5) \in G \times \mathbb{R}^6 \to (g, Q(k), \eta(k, \lambda_1, \dots, \lambda_5)) \in \tilde{Z}.$$

Thus the Liouville form on $\tilde{Z}$ is given by

$$\psi = (1 + mk)\omega + \lambda_1\eta^1 + \lambda_2\eta^2 + \lambda_3\eta^3 + \lambda_4\eta^4 + \lambda_5\eta^5.$$

From the structure (1), we find that

$$\Psi \equiv m\pi \wedge \omega - (1 + m\kappa)\eta^2 \wedge \omega + \sum \mathrm{d}\lambda_\alpha \wedge \eta^\alpha - \lambda_1\pi \wedge \omega + \lambda_2(\kappa\eta^3 - \eta^1) \wedge \omega \\ + \lambda_3(2\eta^2 - \kappa\eta^4) \wedge \omega + \lambda_4(\kappa\eta^5 - \eta^3) \wedge \omega + \lambda_5\eta^4 \wedge \omega,$$

where $\Psi := \mathrm{d}\psi$ and where $\equiv$ denotes equality modulo $\mathrm{span}(\{\eta^\alpha \wedge \eta^\beta\}_{\alpha,\beta=1,\dots,5})$. Let

$$(\partial_\omega, \partial_{\eta^1}, \dots, \partial_{\eta^5}, \partial_{\lambda_1}, \dots, \partial_{\lambda_5}, \partial_\pi)$$

denote the parallelization of $\tilde{Z}$ dual to the coframing

$$(\omega, \eta^1, \dots, \eta^5, \mathrm{d}\lambda_1, \dots, \mathrm{d}\lambda_5, \pi).$$

We then have

$$i_{\partial_{\lambda_i}}\Psi = \eta^i, \quad i = 1, \dots, 5,$$

along with

$$i_{\partial_\omega}\Psi \equiv -\alpha, \qquad i_{\partial_\pi}\Psi \equiv -\beta, \qquad i_{\partial_{\eta^i}}\Psi \equiv -\beta_i, \quad i = 1, \dots, 5,$$

where

$$\alpha = (m - \lambda_1)\,\mathrm{d}k, \tag{9a}$$

$$\beta = (m - \lambda_1)\omega, \tag{9b}$$

$$\beta_1 = \mathrm{d}\lambda_1 + \lambda_2\omega, \tag{9c}$$

$$\beta_2 = \mathrm{d}\lambda_2 + (1 + mk - 2\lambda_3)\omega, \tag{9d}$$

$$\beta_3 = \mathrm{d}\lambda_3 + (\lambda_4 - k\lambda_2)\omega, \tag{9e}$$

$$\beta_4 = \mathrm{d}\lambda_4 + (k\lambda_3 - \lambda_5 - k)\omega, \tag{9f}$$

$$\beta_5 = \mathrm{d}\lambda_5 - k\lambda_4\omega, \tag{9g}$$

and where $\equiv$ denotes equality modulo span$(\eta^1, \dots, \eta^5)$. From these equations, we deduce that the Cartan system $(\mathcal{C}(\Psi), \omega)$ is generated by the differential 1-forms $(\eta^1, \dots, \eta^5, \alpha, \beta, \beta_1, \dots, \beta_5)$.

Theorem 3.19. *The momentum space, Y, is the nine-dimensional sub-manifold of $\tilde{Z}$ defined by the equations*

$$\lambda_2 = \lambda_1 - m = \lambda_3 - \tfrac{1}{2}(1 + mk) = 0,$$

and the Euler–Lagrange system $(\mathcal{E}, \omega)$ is the Pfaffian differential system on Y with independence condition ω generated by the linear differential forms $(\eta^1, \dots, \eta^5, \sigma_1, \sigma_2, \sigma_3)$, where

$$\sigma_1 = \tfrac{1}{2}m\,\mathrm{d}k + \lambda_4\omega, \qquad \sigma_2 = \mathrm{d}\lambda_4 - (\lambda_5 + \tfrac{1}{2}k(1 - mk))\omega, \qquad \sigma_3 = \mathrm{d}\lambda_5 - k\lambda_4\omega.$$

Proof. We let $V_1 \subset T(\tilde{Z})$ be the sub-variety of one-dimensional integral elements of the Cartan system $(\mathcal{C}(\Psi), \omega)$ and denote by $\tilde{Z}_1 \subset \tilde{Z}$ the projection of V_1 under the bundle map $T(\tilde{Z}) \to \tilde{Z}$. From Eqs. (9a) and (9b) we then deduce that $\tilde{Z}_1$ is the submanifold defined by $\lambda_1 = m$. Denote by $\mathcal{C}(\Psi)_1$ the restriction to $\tilde{Z}_1$ of the Cartan system. Then $\mathcal{C}(\Psi)_1$ is generated by the linear differential forms $(\eta^1, \dots, \eta^5, \lambda_2\omega, \beta_2, \dots, \beta_5)$. We then consider the sub-variety $V_2 \subset T(\tilde{Z}_1)$ consisting of integral elements of $(\mathcal{C}(\Psi)_1, \omega)$ and let $\tilde{Z}_2 \subset \tilde{Z}_1$ denote the projection of V_2. We therefore have that $\tilde{Z}_2$ is the sub-manifold of $\tilde{Z}_1$ defined by $\lambda_2 = 0$. Denote by $\mathcal{C}(\Psi)_2$ the restriction to $\tilde{Z}_2$ of $\mathcal{C}(\Psi)_1$. Then $\mathcal{C}(\Psi)_2$ is generated by the linear differential forms $(\eta^1, \dots, \eta^5, (1+mk-2\lambda_3)\omega, \beta_3, \beta_4, \beta_5)$. We proceed as above and let $V_3 \subset T(\tilde{Z}_2)$ be the sub-variety of integral elements of $(\mathcal{C}(\Psi)_2, \omega)$ and define $\tilde{Z}_3 \subset \tilde{Z}_2$ to be the image of V_3 under the projection $T(\tilde{Z}_2) \to \tilde{Z}_2$. It follows that $\tilde{Z}_3$ is the sub-manifold of $\tilde{Z}_2$ defined by the equation $\lambda_3 = (1/2)(1 + mk)$ and that the restriction $\mathcal{C}(\Psi)_3$ of $\mathcal{C}(\Psi)_2$ to $\tilde{Z}_3$ is the Pfaffian differential system generated by $(\eta^1, \dots, \eta^5, \sigma_1, \sigma_2, \sigma_3)$. If we let $V_4 \subset T(\tilde{Z}_3)$ be the set of integral elements of $(\mathcal{C}(\Psi)_3, \omega)$ then the bundle map $V_4 \to \tilde{Z}_3$ is surjective. Hence $Y = \tilde{Z}_3$ and $(\mathcal{C}(\Psi)_3, \omega)$ is the reduced space of $(\mathcal{C}(\Psi), \omega)$. □

Corollary 3.20. *The momentum space Y associated with the Poincaré variational problem for isotropic curves in $\mathbb{R}^{(2,1)}$ is rank 3, affine sub-bundle $Y = G \times \mathcal{F} \subset T^*(G) \cong G \times \mathfrak{g}^*$, where $\mathcal{F} \subset \mathfrak{g}^*$ is defined by*

$$\mathcal{F} = \tfrac{1}{2}(\omega^1 + \omega_1^2) + m\omega_2^1 + \mathrm{span}(\omega_1^2 - \omega^1, \omega^2, \omega^3).$$

The variational problem is non-degenerate, and the characteristic vector field takes the form

$$\xi = \partial_\omega - \frac{2\lambda_4}{m}\partial_k - \lambda_4\partial_{\lambda_3} + \left(\lambda_5 + \frac{1}{2}k(1 - mk)\right)\partial_{\lambda_4} + k\lambda_4\partial_{\lambda_5}. \tag{10}$$

Proof. It follows from the preceding theorem that the restriction of the Liouville to the momentum space takes the form

$$\begin{aligned}\psi_Y &= (1 + mk)\omega + m\eta^1 + \tfrac{1}{2}(1 + mk)\eta^3 + \lambda_4\eta^4 + \lambda_5\eta^5 \\ &= \tfrac{1}{2}(\omega^1 + \omega_1^2) + m\omega_2^1 + \tfrac{1}{2}mk(\omega_1^2 - \omega^1) + \lambda_4\omega^2 + \lambda_5\omega^3.\end{aligned} \tag{11}$$

The form of Y and $\mathcal{F}$ follow directly from this equation. The dimension of Y is equal to 9, and a straightforward calculation shows that

$$\omega \wedge (\Psi_Y)^4 = -12m^2\omega \wedge \mathrm{d}k \wedge \mathrm{d}\lambda_4 \wedge \mathrm{d}\lambda_5 \wedge \eta^1 \wedge \eta^2 \wedge \eta^3 \wedge \eta^4 \wedge \eta^5,$$

which is nowhere vanishing. Hence the variational problem is non-degenerate. The form of the characteristic vector field follows from a direct calculation. □

Remark 3.21. Since the variational problem is non-degenerate, the Euler–Lagrange system $\mathcal{E}$ coincides with the Cartan system of Ψ. The characteristic line-distribution $\Xi \subset T(Y)$ of Ψ is transverse to the independence condition ω, and is generated by the characteristic vector field ξ.

Remark 3.22. Using the explicit form of the Liouville form, we may identify $Y = G \times \mathbb{R}^3$, where $(k, \lambda_4, \lambda_5)$ serve as coordinates on $\mathbb{R}^3$. The explicit form for the characteristic vector field and the 1-forms η^i and ω then imply that the map $\mathcal{H} : Y \to \mathfrak{g}$ is given by

$$\mathcal{H}[\eta(k, \lambda_4, \lambda_5)] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & k & 0 \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathfrak{g}.$$

A smooth map $\Gamma : (a, b) \to Y$ is an integral curve of the Euler–Lagrange system if and only if it satisfies

$$\Gamma^*(\eta^i) = 0, \quad i = 1, \dots, 5, \qquad \Gamma^*(\sigma_i) = 0, \quad i = 1, 2, 3$$

with the independence condition

$$\Gamma^*(\omega) \neq 0.$$

Without loss of generality, we may choose a parameterization of our integral curve such that

$$\Gamma^*(\omega) = \mathrm{d}t.$$

In this case, we may write $\Gamma : t \in (a, b) \to (g(t), k(t), \lambda_4(t), \lambda_5(t)) \in Y = G \times \mathbb{R}^3$. From these relations, and the explicit form of the differential forms η^i and σ_i, we deduce the following result.

Proposition 3.23. *The smooth map $\Gamma : (a, b) \to Y$, parameterized such that $\Gamma^*(\omega) = \mathrm{d}t$, is an integral curve of the Euler–Lagrange system if and only if the real-valued functions $k(t), \lambda_4(t), \lambda_5(t)$ satisfy the relations*

$$\frac{\mathrm{d}k}{\mathrm{d}t} = -\frac{2\lambda_4}{m}, \qquad \frac{\mathrm{d}\lambda_4}{\mathrm{d}t} = \left(\lambda_5 + \frac{1}{2}k(1 - mk)\right), \qquad \frac{\mathrm{d}\lambda_5}{\mathrm{d}t} = k\lambda_4, \tag{12}$$

and $g(t) \in G$ is a solution of

$$g(t)^{-1}\frac{\mathrm{d}g(t)}{\mathrm{d}t} = \mathcal{H}(\eta(t)) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & k(t) & 0 \\ 0 & 1 & 0 & k(t) \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{13}$$

Remark 3.24. Although, in the present case, the Lie group G is not semisimple, it is naturally embedded in $\mathrm{SL}(4, \mathbb{R})$. Using the Killing form on $\mathfrak{sl}(4, \mathbb{R})$ we deduce that the Euler equation (12) may be written in Lax form

$$L' = [L, \mathcal{H}],$$

where

$$L(k, \lambda_4, \lambda_5) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2}(1+mk) & -\lambda_4 & -\lambda_5 & 0 \\ 0 & \frac{1}{2}(1-mk) & 0 & -\lambda_5 \\ -m & 0 & \frac{1}{2}(1-mk) & \lambda_4 \end{pmatrix}.$$

Proposition 3.25. *If $\Gamma : (a, b) \to Y$ is an integral curve of the Euler–Lagrange system, with $\Gamma^*(\omega) = \mathrm{d}t$, then the curvature $k(t)$ satisfies the third-order ordinary differential equation*

$$m\frac{\mathrm{d}^3k}{\mathrm{d}t^3} - 3mk\frac{\mathrm{d}k}{\mathrm{d}t} + \frac{\mathrm{d}k}{\mathrm{d}t} = 0. \tag{14}$$

Conversely, any non-constant solution $k : (a, b) \to \mathbb{R}$ of this equation determines a parameterized integral curve of the Euler–Lagrange system, unique up to the action of $\mathbb{E}(2, 1)$.

Proof. Eq. (14) for $k(t)$ follows directly from (12). Conversely, given a solution of (14), we can uniquely reconstruct $\lambda_4(t), \lambda_5(t)$ from (12), and $g(t)$ is determined, up to initial conditions, by (13). □

Remark 3.26. Under the identification $\mathfrak{g}^* \cong \mathbb{R}^{(2,1)} \oplus \mathbb{R}^{(2,1)}$ introduced in Section 2.4, the Liouville form (11) maps to $(p, v) \in \mathbb{R}^{(2,1)} \oplus \mathbb{R}^{(2,1)}$, where

$$p = -\lambda_5 e_1 + \lambda_4 e_2 - \tfrac{1}{2}(1-mk)e_3, \qquad v = -\tfrac{1}{2}(1+mk)e_1 + me_3,$$

and $\{e_i\}$ is the standard basis of $\mathbb{R}^{(2,1)}$. The Casimir operators therefore take the form

$$C_1 = \|p\|^2 = \lambda_4^2 - \lambda_5(1-mk), \qquad C_2 = \langle p, v\rangle = m\lambda_5 - \tfrac{1}{4}(1-m^2k^2), \tag{15}$$

and are constant along integral curves of the Euler–Lagrange system.

The explicit form of the Casimir operators implies the following result.

Proposition 3.27. *If $\Gamma : (a, b) \to Y$, parameterized such that $\Gamma^*(\omega) = \mathrm{d}t$, is an integral curve of the Euler–Lagrange system then the curvature $k(t)$ satisfies the first-order ordinary differential equation*

$$\left(\frac{\mathrm{d}k}{\mathrm{d}t}\right)^2 = k^3 - \frac{1}{m}k^2 - \frac{1}{m^2}(4C_2 + 1)k + \frac{1}{m^3}(4mC_1 + 4C_2 + 1).$$

Remark 3.28. Letting $h(t) := (1/4)(k - (1/3)m)$, we deduce that h satisfies

$$\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 = 4h^3 - g_2 h - g_3,$$

where

$$g_2 = \frac{1}{m^2}\left(C_2 + \frac{1}{3}\right), \qquad g_3 = \frac{1}{m^3}\left(\frac{mC_1}{4} + \frac{C_2}{6} + \frac{1}{27}\right).$$

Hence the curvature k and the functions λ_4, λ_5 corresponding to any solution of the Euler–Lagrange system may be expressed in terms of Weierstrass elliptic functions with invariants g_2, g_3.

Remark 3.29. A short calculation using the explicit form of $\eta = \psi_Y$ given in (11) and the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$, which can be derived from (2), shows that in the present case the linearized phase portrait $\Pi(\eta) = F(\eta) \cap \mathrm{O}(\eta)$ is one-dimensional, and is spanned by the vector $\mathrm{ad}^*[\mathcal{H}(\eta)]\eta = \lambda_4(\omega_1^2 - \omega^1) - (\lambda_5 + (1/2)k(1 - mk))\omega^2 - k\lambda_4\omega^3$. Hence the regular parts of the phase portraits are one-dimensional in the current problem. We now introduce a more general class of variational problems for which this is the case.

4. Coisotropic variational problems

Consider a smooth manifold M equipped with an exterior differential 2-form Ψ. The kernel of Ψ_x will be denoted by $N(\Psi)_x$. Suppose that a Lie group G acts on M. Denote by $A^\sharp$ the fundamental vector field on M corresponding to $A \in \mathfrak{g}$ and, for each $x \in M$, let $\mathfrak{g}^\sharp(M)_x \subset T_x(M)$ be the vector subspace $\{A_x^\sharp : A \in \mathfrak{g}\}$. We denote by $\mathfrak{g}^\sharp(M)_x^\perp$ the *polar space* of $\mathfrak{g}^\sharp(M)_x$ with respect to Ψ_x:

$$\mathfrak{g}^\sharp(M)_x^\perp := \{v \in T_x(M) : \Psi_x(v, A^\sharp) = 0,\ \forall A^\sharp \in \mathfrak{g}^\sharp(M)_x\}.$$

Definition 4.1. The action of G on M is *coisotropic* with respect to Ψ if

$$\mathfrak{g}^\sharp(M)_x^\perp \subset \mathfrak{g}^\sharp(M)_x + N(\Psi)_x \quad \forall x \in M.$$

Remark 4.2. The notion of a coisotropic action arises naturally when studying collective complete integrability of Hamiltonian systems (see [13,14,25]).

Definition 4.3. An invariant Lagrangian $L : \mathbb{A} \to \mathbb{R}$ is said to be *coisotropic* if it is non-degenerate and if the action of G on the regular part of the momentum space Y is coisotropic with respect to Ψ_Y, the restriction of the canonical 2-form Ψ to Y.

(Recall that we are now using the notation Y, $\mathcal{F}$ and $\mathcal{P}(\mu)$ to denote the regular parts of the momentum space, phase space and phase portraits, respectively.)

Proposition 4.4. *A non-degenerate invariant Lagrangian* $L : \mathbb{A} \to \mathbb{R}$ *is coisotropic if and only if the linearized phase portrait* $\Pi(\eta)$ *is spanned by* $\mathrm{ad}^*[\mathcal{H}(\eta)]\eta$, *for every* $\eta \in \mathcal{F}$.

Proof. Using the left-invariant trivialization, we find that the polar space of $\mathfrak{g}^\sharp(Y)_{(g,\eta)}$ is given by

$$\mathfrak{g}(\eta)^\perp := \mathfrak{g}^\sharp(Y)^\perp_{(g,\eta)} = \{A + V \in \mathfrak{g} \oplus F(\eta) : V = -\mathrm{ad}^*(A)\eta\}.$$

First, assume that L is coisotropic, i.e.

$$\mathfrak{g}(\eta)^\perp \subset \mathfrak{g} + \mathrm{span}(\mathcal{H}(\eta) - \mathrm{ad}^*[\mathcal{H}(\eta)]\eta).$$

Let $V \in \Pi(\eta)$. Then $V \in F(\eta)$ and there exists an $A \in \mathfrak{g}$ such that $V = -\mathrm{ad}^*(A)\eta$. Then $A + V$ belongs to $\mathfrak{g}(\eta)^\perp$ and hence V must be a real multiple of $\mathrm{ad}^*[\mathcal{H}(\eta)]\eta$.

Conversely, assume that $\Pi(\eta)$ is spanned by $\mathrm{ad}^*[\mathcal{H}(\eta)]\eta$. Given any element, $A + V$, of the polar space $\mathfrak{g}(\eta)^\perp$, then $V \in \Pi(\eta)$ so there exists $s \in \mathbb{R}$ such that $V = s \cdot \mathrm{ad}^*[\mathcal{H}(\eta)]\eta$. Therefore, we can write

$$A + V = -s(\mathcal{H}(\eta) - \mathrm{ad}^*[\mathcal{H}(\eta)]\eta) + (A + s\mathcal{H}(\eta)).$$

Since $A + s\mathcal{H}(\eta)$ is an element of $\mathfrak{g}$, it follows that $A + V \in \mathfrak{g} + \mathrm{span}(\mathcal{H}(\eta) - \mathrm{ad}^*[\mathcal{H}(\eta)]\eta)$. Therefore $\mathfrak{g}(\eta)^\perp \subset \mathfrak{g} + \mathrm{span}(\mathcal{H}(\eta) - \mathrm{ad}^*[\mathcal{H}(\eta)]\eta)$, as required. □

Remark 4.5. Note that if $\eta \in \mathcal{F}$ then $\Pi(\eta)$ is one-dimensional and the map $\rho(\eta) : \mathrm{Ann}(F(\eta)) \to \mathrm{O}(\eta)$ defined in (6) is injective.

Proposition 4.6. *Let* $L : \mathbb{A} \to \mathbb{R}$ *be an invariant coisotropic Lagrangian and let* $Y = G \times \mathcal{F}$ *be the corresponding momentum space. Suppose that* $\mathcal{F}$ *is non-empty, we then have*:

- $\dim(Y) = \dim(G) + \mathrm{rank}(G) + 1$;
- *the regular part of the phase space,* $\mathcal{F}$, *intersects the coadjoint orbits transversally*;
- *the regular parts* $\mathcal{P}(\mu)$ *of the phase portraits are smooth and one-dimensional*;
- *every* $\eta \in \mathcal{F}$ *is a regular element of* $\mathfrak{g}^*$. *In particular, the isotropy group* G_η *and the isotropy algebra* $\mathfrak{g}_\eta$ *are Abelian.*

Proof. For each $\eta \in \mathcal{F}$, let $k(\eta)$ be the dimension of the isotropy Lie algebra $\mathfrak{g}_\eta$. Note that

$$\dim(F(\eta) \cap \mathrm{O}(\eta)) = 1, \qquad \dim(\mathrm{O}(\eta)) = \dim(G) - k(\eta),$$
$$\dim(\mathrm{Ann}(F(\eta))) = \dim(G) - \dim(F(\eta)).$$

We then have

$$\dim(F(\eta)) + \dim(\mathrm{O}(\eta)) - 1 \leq \dim(G),$$

which in turn implies that $\dim(\mathcal{F}) \le k(\eta) + 1$. On the other hand, from the injectivity of the map $\rho(\eta) : \mathrm{Ann}(F(\eta)) \to \mathrm{O}(\eta)$, it follows that $k(\eta) \le \dim(\mathcal{F})$. Therefore we have

$$k(\eta) \le \dim(\mathcal{F}) \le k(\eta) + 1.$$

Notice that $\dim(G) + k(\eta)$ is even and that $\dim(Y) = \dim(G) + \dim(\mathcal{F})$ is odd. Thus, we must have $k(\eta) + 1 = \dim(\mathcal{F})$. In particular, $k(\eta) = k$ is constant and

$$\dim(Y) = \dim(G) + k + 1.$$

This implies that

$$\dim(\mathfrak{g}) = \dim(F(\eta)) + \dim(\mathrm{O}(\eta)) - 1.$$

Thus $\mathcal{F}$ intersects the coadjoint orbits transversally. Since $\dim(F(\eta) \cap \mathcal{O}(\eta)) = 1$, it follows that $\mathcal{P}(\eta) = \mathcal{F} \cap \mathcal{O}(\eta)$ is a smooth curve such that $T_\eta[\mathcal{P}(\eta)] = \Pi(\eta)$. Moreover, from the transversality condition, it follows that $\mathcal{F}$ cannot be contained in the set $\mathfrak{g}^*_{\mathrm{s}}$ of the singular element of $\mathfrak{g}^*$. Thus, $\mathcal{F} \cap \mathfrak{g}^*_{\mathrm{r}}$ is non-empty. Therefore, there exists an $\eta \in \mathcal{F}$ such that $k = k(\eta) = \mathrm{rank}(G)$. This gives the required result. □

Remark 4.7. The regular part of the phase space, $\mathcal{F}$, is foliated by the nowhere vanishing vector field Φ_ξ and the leaves are the phase portraits. Furthermore, if $X \subset \mathcal{F}$ is a local section of such a foliation then X is also a local section of the coadjoint representation.

Definition 4.8. Let $L : \mathbb{A} \to \mathbb{R}$ be a coisotropic Lagrangian. The *moment map* $J : Y \to \mathfrak{g}^*$ of the Hamiltonian action of G on Y is defined by

$$J(g, \eta) = \mathrm{Ad}^*(g)\eta \quad \forall (g, \eta) \in Y.$$

Proposition 4.9. *Let $L : \mathbb{A} \to \mathbb{R}$ be a coisotropic Lagrangian. Then*:

- $J(Y) \subset \mathfrak{g}^*_{\mathrm{r}}$;
- $J : Y \to \mathfrak{g}^*$ *is a submersion*;
- $J^{-1}(\mu)$ *is a* $(k + 1)$*-dimensional submanifold of* Y *such that*

  $$T_{(g,\eta)}[J^{-1}(\mu)] = \ker[\mathrm{d}J|_{(g,\eta)}] = \mathrm{span}[\xi|_{(g,\eta)}] + \mathfrak{g}_\eta.$$

  so that the characteristic vector field $\xi|_{J^{-1}(\mu)}$ *is tangent of* $J^{-1}(\mu)$, *and* G_μ *acts freely and properly on* $J^{-1}(\mu)$.
- $Y_\mu := J^{-1}(\mu)/G_\mu$ *is a one-dimensional manifold and* $J^{-1}(\mu) \to Y_\mu$ *is a principal* G_μ *bundle*.
- $Y_\mu \cong \mathcal{P}(\mu)$, *the phase portrait*.

Proof. From Proposition 4.6 we know that each $\eta \in \mathcal{F}$ is an element of $\mathfrak{g}^*_{\mathrm{r}}$, and hence $J(g, \eta) \in \mathfrak{g}^*_{\mathrm{r}}$ for all $(g, \eta) \in Y$. The differential of the moment map is given by the formula

$$\langle \mathrm{d}J|_{(g,\eta)}(A + V); B\rangle = \langle \mathrm{ad}^*(A)\eta + V; B\rangle \quad \forall A + V \in \mathfrak{g} \oplus F(\eta), \ \forall B \in \mathfrak{g}. \tag{16}$$

This implies that

$$\mathrm{Im}[\mathrm{d}J|_{(g,\eta)}] = F(\eta) + \mathrm{O}(\eta)$$

for all $\eta \in \mathcal{F}$. Since $\mathcal{F}$ and $\mathcal{O}(\eta)$ intersect transversally, this implies that J is a submersion. Therefore $J^{-1}(\mu)$ is a sub-manifold of Y and the tangent space $T_{(g,\eta)}[J^{-1}(\mu)]$ is naturally isomorphic to $\ker[\mathrm{d}J|_{(g,\eta)}]$. From the formula (16) and the fact that the Lagrangian is coisotropic, we deduce that

$$\ker[\mathrm{d}J|_{(g,\eta)}] = \mathrm{span}[\xi|_{(g,\eta)}] + \mathfrak{g}_\eta$$

for all $\eta \in \mathcal{F}$, as required. Note that this relation implies that the characteristic vector field ξ belongs to $\ker[\mathrm{d}J]$, and therefore that $\xi|_{(g,\eta)}$ is tangent to $J^{-1}(\mu)$. We shall denote the restriction of ξ to the fiber $J^{-1}(\mu)$ by ξ^μ.

The isotropy group G_μ acts on $J^{-1}(\mu)$ by $(g,\eta) \mapsto (hg,\eta)$ for each $h \in G_\mu$. This action is clearly free and proper, so the quotient space $Y_\mu := J^{-1}(\mu)/G_\mu$ exists as a one-dimensional manifold. The map

$$\pi_\mu : (g,\eta) \in J^{-1}(\mu) \to [(g,\eta)] \in Y_\mu$$

gives $J^{-1}(\mu)$ the structure of a principal fiber-bundle with structure group G_μ. Moreover, the vector field ξ^μ is horizontal with respect to the fibration $J^{-1}(\mu) \to Y_\mu$.

We also consider the fibration of $J^{-1}(\mu)$ over $\mathcal{P}(\mu)$ defined by

$$\tilde{\pi}_\mu : (g,\eta) \in J^{-1}(\mu) \to \eta \in \mathcal{P}(\mu).$$

The structure group is again the isotropy subgroup G_μ. Furthermore, $\tilde{\pi}_\mu$ is constant along the fibers of the fibration π_μ, and therefore descends to a diffeomorphism of Y_μ onto $\mathcal{P}(\mu)$. □

Definition 4.10. We adopt standard terminology, referring to Y_μ as the *Marsden–Weinstein reduction* of Y at μ, and to $\pi_\mu : J^{-1}(\mu) \to Y_\mu$ as the *Marsden–Weinstein fibration* at μ. We may consider $\pi_\mu : J^{-1}(\mu) \to Y_\mu$ as a principal G_μ bundle over Y_μ, where the right-action of G_μ on $J^{-1}(\mu)$ is given by $R_h(g,\eta) = (hg,\eta)$, for all $h \in G_\mu$, for $(g,\eta) \in J^{-1}(\mu)$.

Definition 4.11. Let $\mu \in J(Y)$. The restriction of the Marsden–Weinstein fibration $J^{-1}(\mu)$ to the reduced phase portrait $\tilde{\mathcal{P}}(\mu)$ is said to be the *reduced Marsden–Weinstein fibration*. We shall denote this fibration by $\tilde{\pi}_\mu : P^\mu \to \tilde{\mathcal{P}}(\mu)$.

Remark 4.12. The vector field $\Phi_\xi : \eta \mapsto -\mathrm{ad}^*[\mathcal{H}(\eta)]\eta$ is tangent to the phase portraits. We denote by Φ^μ_ξ the restriction of Φ_ξ to $\tilde{\mathcal{P}}(\mu)$. Note that the vector fields ξ^μ and Φ^μ_ξ are related by the fibration $\tilde{\pi}_\mu : P^\mu \to \tilde{\mathcal{P}}(\mu)$.

Remark 4.13. On the reduced phase portrait $\tilde{\mathcal{P}}(\mu)$ there exists a unique nowhere vanishing 1-form σ^μ such that $\sigma^\mu(\Phi^\mu_\xi) = 1$. Take $\eta \in \tilde{\mathcal{P}}(\mu)$, then the integral curve $\phi_\eta : (\epsilon^-(\eta), \epsilon^+(\eta)) \to \mathfrak{g}^*$ is a maximal parameterization of $\tilde{\mathcal{P}}(\mu)$ such that $\phi^*_\eta(\sigma^\mu) = \mathrm{d}t$.

Definition 4.14. On P^μ we consider the $\mathfrak{g}_\mu$-valued 1-form θ^μ defined by

$$\theta^\mu|_{(g,\eta)} := \mathrm{Ad}(g)(\Theta - \mathcal{H}\sigma^\mu).$$

This defines a connection on the reduced Marsden–Weinstein fibration $P^\mu \to \tilde{\mathcal{P}}(\mu)$. We call θ^μ the *canonical connection* of the reduced Marsden–Weinstein fibration $P^\mu \to \tilde{\mathcal{P}}(\mu)$.

4.1. Isotropic curves in $\mathbb{R}^{(2,1)}$

In the case of our problem for isotropic curves in $\mathbb{R}^{(2,1)}$, we have defined a map

$$\mathbb{R}^3 \hookrightarrow \mathfrak{g}^* \cong \mathbb{R}^{(2,1)} \oplus \mathbb{R}^{(2,1)}, \qquad y = (k, \lambda_4, \lambda_5) \mapsto (p(y), v(y)), \tag{17}$$

where

$$p = \begin{pmatrix} -\lambda_5 \\ \lambda_4 \\ -\frac{1}{2}(1 - mk) \end{pmatrix}, \qquad v = \begin{pmatrix} -\frac{1}{2}(1 + mk) \\ 0 \\ m \end{pmatrix}.$$

Given the form (10) of the characteristic vector field ξ, we see that the regular part of the phase space $\mathcal{F}$ is given by the complement of the set of points with

$$\lambda_4 = 0, \qquad \lambda_5 + \tfrac{1}{2}k(1 - mk) = 0. \tag{18}$$

To show that the action of $\mathbb{E}(2, 1)$ on the regular part of the momentum space, Y, is coisotropic, we consider a general vector field on Y:

$$Z^1\frac{\partial}{\partial k} + Z^2\frac{\partial}{\partial\lambda_4} + Z^4\frac{\partial}{\partial\lambda_5} + X^i\frac{\partial}{\partial\omega^i} + Y^1\frac{\partial}{\partial\omega_1^1} + Y^2\frac{\partial}{\partial\omega_1^2} + Y^3\frac{\partial}{\partial\omega_2^1}.$$

This vector field lies in $\mathfrak{g}(Y)^\perp|_{(g,k,\lambda_4,\lambda_5)}$ if and only if

$$\tfrac{1}{2}mZ^1 = -\tfrac{1}{2}(1 - mk)Y^1 - \lambda_4 Y^2, \tag{19a}$$

$$Z^2 = \tfrac{1}{2}(1 - mk)Y^3 + \lambda_5 Y^2, \tag{19b}$$

$$Z^3 = \lambda_4 Y^3 - \lambda_5 Y^1, \tag{19c}$$

and

$$-\tfrac{1}{2}(1 - mk)X^1 + \lambda_5 X^3 - mY^3 + \tfrac{1}{2}(1 + mk)Y^2 = 0, \tag{20a}$$

$$-\lambda_4 X^1 - \lambda_5 X^2 - \tfrac{1}{2}mZ^1 - \tfrac{1}{2}(1 + mk)Y^1 = 0, \tag{20b}$$

$$-\tfrac{1}{2}(1 - mk)X^2 - \lambda_1 X^3 + mY^1 = 0. \tag{20c}$$

If $(k, \lambda_4, \lambda_5)$ lies in the regular part of $\mathcal{F}$ then conditions (19) and (20) are linearly independent. Therefore in this case

$$\dim \mathfrak{g}(Y)^\perp = 3.$$

It is easily checked that the characteristic vector field ξ belongs to $\mathfrak{g}(Y)^{\perp}$, as do the following vector fields:

$$S_1 = -\lambda_4 \frac{\partial}{\partial \omega_1^1} + \frac{1}{2}(1-mk)\frac{\partial}{\partial \omega_1^2} - \lambda_5 \frac{\partial}{\partial \omega_2^1} - m\frac{\partial}{\partial \omega^3},$$
$$S_2 = \lambda_5 \frac{\partial}{\partial \omega^1} - \lambda_4 \frac{\partial}{\partial \omega^2} + \frac{1}{2}(1-mk)\frac{\partial}{\partial \omega^3}.$$

Hence we have that

$$\mathfrak{g}(Y)^{\perp} = \mathrm{span}(\xi, S_1, S_2) \subset \mathrm{span}(\xi) \oplus \mathfrak{g}.$$

Hence the action is coisotropic.

The momentum map and the basic invariants. From the form of the coadjoint action of $\mathbb{E}(2,1)$ given earlier, we deduce that the moment map takes the form

$$J(g; y) = (Ap(y), Av(y) - (Ap(y)) \times Q)$$

for all $g = (Q, A) \in \mathbb{E}(2,1)$. Note that $J(Y) \subseteq \mathfrak{g}_{\mathrm{r}}^*$. The basic invariants, which correspond to constants of motion of the system, are the Casimir operators

$$C_1(g, y) := \|p(y)\|^2 = \lambda_4^2 - \lambda_5(1-mk),$$
$$C_2(g, y) := \langle p(y), v(y)\rangle = m\lambda_5 - \tfrac{1}{2}(1-m^2k^2).$$

If we choose $\mu = (\mathfrak{m}_1, \mathfrak{m}_2) \in \mathfrak{g}_{\mathrm{r}}^*$, where $\mathfrak{m}_1, \mathfrak{m}_1 \in \mathbb{R}^{(2,1)}$, then $J^{-1}(\mu)$ is the set

$$\lambda_4^2 = \frac{m^2}{4}k^3 - \frac{m}{4}k^2 - \left(\frac{1}{4} + \langle \mathfrak{m}_1, \mathfrak{m}_2\rangle\right)k + \left(\|\mathfrak{m}_1\|^2 + \frac{1}{4m} + \frac{\langle \mathfrak{m}_1, \mathfrak{m}_2\rangle}{4}\right), \tag{21a}$$

$$\lambda_5 = \frac{1}{m}\left(\frac{1}{4}(1-m^2k^2) + \langle \mathfrak{m}_1, \mathfrak{m}_2\rangle\right). \tag{21b}$$

If we perform the substitution

$$k = \left(\frac{4}{m}\right)^{2/3}\chi + \frac{1}{3m}, \tag{22}$$

then the first equation becomes the cubic relation

$$\lambda_4^2 = 4\chi^3 - g_2\chi - g_3, \tag{23}$$

where we have defined the “modified Casimirs”

$$g_2(\mathfrak{m}_1, \mathfrak{m}_2) = \left(\frac{4}{m}\right)^{2/3}\left(\frac{1}{3} + \langle \mathfrak{m}_1, \mathfrak{m}_2\rangle\right),$$
$$g_3(\mathfrak{m}_1, \mathfrak{m}_2) = -\left(\|\mathfrak{m}_1\|^2 + \frac{2}{3m}\langle \mathfrak{m}_1, \mathfrak{m}_2\rangle + \frac{4}{27m}\right).$$

Parameterization of the phase portraits. We define the discriminant of the cubic polynomial appearing in Eq. (23)

$$D(m_1, m_2) = 27g_3^2 - g_2^3.$$

There are two non-degenerate cases that we must consider.

Case I: $D(m_1, m_2) > 0$. In this case the cubic polynomial has one real root and two complex-conjugate roots. We may parameterize the curve by taking $\chi(t) = \wp(t; g_2, g_3)$ with $t \in (0, 2\omega_1)$, where ω_1, ω_2 and $\omega_3 = (1/2)(\omega_1 + \omega_2)$ are the half-periods of the $\wp$ function. From (22), we then find that k may be written in terms of elliptic functions and, solving (21) then gives λ_4 and λ_5 in terms of elliptic functions.

Case II: $D(m_1, m_2) < 0$. In this case the cubic polynomial has three distinct real roots, and the curve (23) has two disjoint components. The “compact” component may be parameterized by $\chi(t) = \wp_3(t; g_2, g_3) := \wp(t + \omega_3; g_2, g_3)$ with $t \in \mathbb{R}$. This solution is periodic, with period $2\omega_1$. The “unbounded” component of the curve is parameterized by $\chi(t) = \wp(t; g_2, g_3)$, $t \in (0, 2\omega_1)$.

In the degenerate cases where the discriminant vanishes at $D = 0$, the cubic is singular, and our curve is rational. In this case, the $\wp$ and $\wp_3$ functions degenerate into elementary functions.

5. Integrability by quadratures

Proposition 5.1. *The integral curves of the characteristic vector field ξ with momentum $\mu \in J(Y)$ are the horizontal curves of the canonical connection θ^μ on P^μ.*

Proof. Consider a horizontal curve $\Gamma : (a, b) \to P^\mu$ of the canonical connection. We write $\Gamma(t) = (h(t), \eta(t))$, where $h : (a, b) \to G$ and $\eta : (a, b) \to \mathcal{P}(\mu)$ are smooth curves. Without loss of generality we can assume that $\eta^*(\sigma^\mu) = \mathrm{d}t$, so that

$$\eta'(t) = \Phi^\mu_\xi|_{\eta(t)} = -\mathrm{ad}^*[\mathcal{H}(\eta(t))]\eta(t) \quad \forall t \in (a, b).$$

Since Γ is horizontal, we then have

$$0 = \Gamma^*(\theta^\mu) = g(t)(h^{-1}(t)h'(t) - \mathcal{H}[\eta(t)])g(t)^{-1}\,\mathrm{d}t.$$

This implies that $\Gamma'(t) = \xi|_{\Gamma(t)}$, for all $t \in (a, b)$.

Conversely, if $\Gamma : (a, b) \to Y$ is an integral curve of ξ with momentum μ then $\Gamma(t) \in P^\mu$, for all $t \in (a, b)$. Furthermore, we know that

$$\Gamma'(t) = \mathcal{H}(\eta(t)) - \mathrm{ad}^*[\mathcal{H}(\eta(t))]\eta(t).$$

Thus $h^{-1}(t)h'(t) = \mathcal{H}[\eta(t)]$ and hence $\Gamma^*[\theta^\mu] = 0$. □

Since the structure group of the Marsden–Weinstein fibrations is Abelian and the base manifolds are one-dimensional, the horizontal curves of the canonical connection can be found by a single quadrature. The explicit integration of the horizontal curves requires four steps:

- Step 1: take a smooth parameterization $\eta : (a, b) \to \tilde{\mathcal{P}}(\mu)$ of the phase portrait.
- Step 2: compute $v^\mu : (a, b) \to \mathbb{R}$ such that $\eta^*(\sigma^\mu) = v^\mu \, \mathrm{d}t$.
- Step 3: take any map $g : (a, b) \to G$ such that $(g^{-1}, \eta) : (a, b) \to Y$ is a cross-section of the reduced Marsden–Weinstein fibration $\tilde{\pi}_\mu : P^\mu \to \tilde{\mathcal{P}}(\mu)$. This involves solving the equation

$$\mathrm{Ad}^*(g(t)^{-1})\eta(t) = \mu$$

  for $g(t)$.
- Step 4: compute the gauge transformation

$$h(t) = \exp\left[\int_{t_0}^{t} (g^{-1}(u)\mathcal{H}[\eta(u)]g(u)v^\mu(u) + g^{-1}(u)g'(u))\, \mathrm{d}u\right] \quad \forall t \in (a, b). \tag{24}$$

  Note that the fact that $(g(t)^{-1}, \eta(t))$ is a section of $\tilde{\pi}_\mu : P^\mu \to \tilde{\mathcal{P}}(\mu)$, along with the definitions of $\mathcal{H}[\eta]$ and v^μ imply that $g^{-1}(t)\mathcal{H}[\eta(t)]g(t)v^\mu(t) + g^{-1}(t)g'(t) \in \mathfrak{g}_\mu$ for all $t \in (a, b)$. Hence $h(t) \in G_\mu$, for all $t \in (a, b)$.

  Conclusion. The image of the curve $\Gamma : (a, b) \to Y$ defined by

$$\Gamma(t) = (h(t)g(t)^{-1}, \eta(t)) \quad \forall t \in (a, b) \tag{25}$$

is contained in P^μ and Γ is horizontal for the canonical connection θ^μ. Any horizontal curve of P^μ arises in this way.

Remark 5.2. In the case where the symmetry group G is a classical matrix group, with $G \subset \mathrm{Aut}(V)$ for V a finite-dimensional vector space, and the Lagrangian is a polynomial function, then we have:

- The phase space $\mathcal{F}$ is an algebraic subset of $\mathfrak{g}^*$.
- The generic coadjoint orbit is defined by polynomial equations $F_j(\eta) = 0$, $j = 1, \ldots, k$, where k is the rank of $\mathfrak{g}$ and where $F_1, \ldots, F_k$ is a basis of the Ad^*-invariant polynomial functions $\mathfrak{g}^* \to \mathbb{R}$.
- The phase portraits are real-algebraic curves.

In the simplest cases the phase portraits are rational or elliptic curves, so they can be easily parameterized by means of elementary or elliptic functions (see the examples considered in Refs. [5,12,16,20–23]). The third step in the construction above can be treated in a rather easy way if $\mathfrak{g}$ is semisimple and if the momentum μ is a regular semisimple element of $\mathfrak{g}^*$. In this case the construction of a cross-section of the Marsden–Weinstein fibration is a linear-algebra problem involving the structure of the Cartan subalgebras of $\mathfrak{g}$.

Definition 5.3. Consider $\mu \in J(Y)$. We say that μ is a *complete momentum* if $\epsilon^-(\eta) = -\infty$ and $\epsilon^+(\eta) = \infty$ for some (and hence for all) $\eta \in \tilde{\mathcal{P}}(\mu)$.

Proposition 5.4. *If $\mu \in J(Y)$ is a complete momentum, then the connected components of the reduced Marsden–Weinstein fibration $P^\mu \to \tilde{\mathcal{P}}(\mu)$ are Euclidean cylinders and ξ^μ is a linear vector field.*

Proof. Let $Q(\mu)$ be a connected component of P^μ and let $\eta : \mathbb{R} \to \tilde{P}(\mu)$ be an integral curve of the phase flow that parameterizes the reduced phase portrait. Since $\mathbb{R}$ is contractible, $Q(\mu) \to \tilde{P}(\mu)$ is a trivial fiber bundle. This implies that there exists a smooth map $g : \mathbb{R} \to G$ such that $(g^{-1}, \eta) : \mathbb{R} \to G \times \tilde{P}(\mu)$ is a cross-section of $Q(\mu) \to \tilde{P}(\mu)$. Fix $(g_0, \eta_0) \in Q(\mu)$ and let $t_0 \in \mathbb{R}$ such that $\eta(t_0) = \eta_0$ and consider the curve $\Gamma_{(g_0,\eta_0)} : \mathbb{R} \to Q(\mu)$ defined by

$$\Gamma_{(g_0,\eta_0)}(t) = (g_0 g(t_0)k(t), \eta(t)) \quad \forall t \in \mathbb{R},$$

where

$$k(t) = \exp\left[\int_{t_0}^{t} (g(u)^{-1}\mathcal{H}[\eta(u)]g(u) + g(u)^{-1}g'(u))\,\mathrm{d}u\right] g(t)^{-1} \quad \forall t \in \mathbb{R}.$$

Then, $\Gamma_{(g_0,\eta_0)}$ is the integral curve of ξ^μ with initial condition $\Gamma_{(g_0,\eta_0)}(t_0) = (g_0, \eta_0)$. This shows that the restriction of the vector field ξ^μ to $Q(\mu)$ is complete. Now fix a basis $(e_1, \ldots, e_k)$ of $\mathfrak{g}_\mu$ and let $e_1^\sharp, \ldots, e_k^\sharp$ denote the corresponding fundamental vector fields on $Q(\mu)$. Then $\{\xi^\mu, e_1^\sharp, \ldots, e_k^\sharp\}$ is a set of complete, linearly independent and commuting vector fields on Q^μ. It is then a standard fact that $Q(\mu)$ is a $(k+1)$-dimensional cylinder, that is $Q(\mu) = \mathbb{R}^{k+1}/K$, where $K \subset \mathbb{R}^{k+1}$ is a subgroup of $\mathbb{R}^{k+1}$ generated over $\mathbb{Z}$ by $m \leq k+1$ linearly independent vectors $a_1, \ldots, a_m$:

$$K = \left\{ \sum_{j=1}^{m} n_j a_j : n_j \in \mathbb{Z} \right\}.$$

The vector fields $\{\xi^\mu, e_1^\sharp, \ldots, e_k^\sharp\}$ are then the push-forward of linear vector fields $b_0, b_1, \ldots, b_k$ on $\mathbb{R}^{k+1}$. This yields the required result. □

Remark 5.5. If $\tilde{\mathcal{P}}(\mu)$ is compact and the isotropy subgroup G_μ is compact, then the connected components of the reduced Marsden–Weinstein fibration P^μ are $(k+1)$-dimensional tori.

5.1. Cross-sections of the Marsden–Weinstein fibration for isotropic curves in $\mathbb{R}^{(2,1)}$

Finally, we show how the above integration procedure may be carried out in our example.

Given $y = (k, \lambda_4, \lambda_5) \in \mathbb{R}^3$, we have defined the vectors $(p, v) \in \mathbb{R}^{(2,1)} \oplus \mathbb{R}^{(2,1)} \cong \mathfrak{g}^*$. With respect to the standard basis (e_1, e_2, e_3) for $\mathbb{R}^{(2,1)}$ these take the form

$$p = \sum_{i=1}^{3} p^i e_i := -\lambda_5 e_1 + \lambda_4 e_2 - \frac{1}{2}(1 - mk)e_3,$$

$$v = \sum_{i=1}^{3} v^i e_i := -\frac{1}{2}(1 + mk)e_1 + m e_3.$$

Letting $\mu = (\mathfrak{m}_1, \mathfrak{m}_2) \in \mathrm{Im}\, J \subseteq \mathfrak{g}_{\mathrm{r}}^*$, we wish to construct the map $g : (a, b) \subset \mathbb{R} \to G$ with the property that (g^{-1}, η) is a section of the reduced Marsden–Weinstein fibration $\tilde{\pi}_\mu : P^\mu \to \tilde{\mathcal{P}}(\mu)$. We must consider separately the cases where the coadjoint orbit is of positive, negative or null type.

Positive type. For an orbit of positive type, where $C_1 = \|p\|^2 > 0$, we may assume, up to the action of G on $\mathcal{O}(\mu)$, that $\mu = (\mathfrak{m}_1, \mathfrak{m}_2)$ is in the standard form:

$$\mathfrak{m}_1 = \sqrt{C_1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathfrak{m}_2 = \frac{C_2}{\sqrt{C_1}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

where $C_2 := \langle p, v \rangle$.

We now wish to construct $g = (Q, A) \in \mathbb{E}(2, 1)$ with the property that

$$\eta = (p, v) = \mathrm{Ad}^*(g)\mu.$$

Since $\|p\|^2 > 0$, we may define the $\mathbb{R}^{(2,1)}$-valued map

$$A_2 := \frac{p}{\sqrt{C_1}} = \frac{p}{\|p\|}$$

with the property that $\|A_2\|^2 = 1$. We can now complete A_2 to a frame field (A_1, A_2, A_3) by adding any $\mathbb{R}^{(2,1)}$-valued functions A_1, A_3 with the property that

$$\langle A_i, A_j \rangle = g_{ij}, \quad i, j = 1, 2, 3,$$

and we fix the orientation of this basis by the requirements that

$$A_2 \times A_1 = A_1, \qquad A_2 \times A_3 = -A_3, \qquad A_3 \times A_1 = A_2.$$

More explicitly, we can define the vector $S = \lambda_4 e_1 + (\lambda_5 - (1/2)(1 - mk))e_2 - \lambda_4 e_3$, with the property that $(p, S) = 0$. We then define $A = (A_1, A_2, A_3) : P^\mu \to \mathrm{SO}(2, 1)$ by

$$A_1 = \frac{1}{\sqrt{2}} \left(\frac{p}{\|p\|} \times \frac{S}{\|S\|} + \frac{S}{\|S\|} \right), \qquad A_2 = \frac{p}{\|p\|},$$
$$A_3 = \frac{1}{\sqrt{2}} \left(\frac{p}{\|p\|} \times \frac{S}{\|S\|} - \frac{S}{\|S\|} \right).$$

Defining the map $Q : P^\mu \to \mathbb{R}^{(2,1)}$ by

$$Q = -\frac{1}{\sqrt{C_1}} A_2 \times v = \frac{\langle v, A_3 \rangle}{\|p\|^2} A_1 - \frac{\langle v, A_1 \rangle}{\|p\|^2} A_3,$$

we let

$$g := (Q, A) : P^\mu \to \tilde{\mathcal{P}}(\mu).$$

It then follows that $(p, v) = \mathrm{Ad}^*(g)\mu$, as required. Therefore the map $(p, v) \in \tilde{\mathcal{P}}(\mu) \to (g(p, v)^{-1}, (p, v)) \in P^\mu$ is a cross-section of the Marsden–Weinstein fibration.

Negative type. For orbits of negative type we have $C_1 := \|p\|^2 < 0$. We treat the case where the vector p is future-directed, although the past-directed case may be treated similarly. The standard form of elements in this case is $\mu = (\mathfrak{m}_1, \mathfrak{m}_2)$, where

$$\mathfrak{m}_1 = \sqrt{\frac{|C_1|}{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \qquad \mathfrak{m}_2 = -\frac{C_2}{\sqrt{2|C_1|}}\begin{pmatrix}1\\0\\1\end{pmatrix}.$$

To define a suitable basis, we fix a vector $S \in \mathbb{R}^{(2,1)}$ with $\|S\|^2 = 1$ and $\langle p, S\rangle = 0$. (For example, the vector $S = e_2 + p^2/p^3 e_1$.) We then define a basis

$$A_1 = \frac{1}{\sqrt{2|C_1|}}(p - p \times S), \qquad A_2 = S, \qquad A_3 = \frac{1}{\sqrt{2|C_1|}}(p + p \times S).$$

Letting $A := (A_1, A_2, A_3) \in \mathrm{SO}(2, 1)$, and

$$Q := \frac{1}{|C_1|} p \times v = \frac{1}{\sqrt{2|C_1|}}(\langle v, A_1 - A_3\rangle A_2 - \langle v, A_2\rangle(A_1 - A_3)),$$

we then define $g := (Q, A)$. It follows that $\eta = \mathrm{Ad}^*(g)\mu$, and hence that the map $(p, v) \mapsto (g(p, v)^{-1}, (p, v))$ is a cross-section of the Marsden–Weinstein fibration.

Null type. Finally, orbits of negative type have $C_1 := \|p\|^2 = 0$ with $p \neq 0$. Again we treat the case where the vector p is future-directed, the past-directed case being similar. In this case, we may use the action of $\mathbb{E}(2, 1)$ to reduce $\mu = (\mathfrak{m}_1, \mathfrak{m}_2)$ to the standard form

$$\mathfrak{m}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad \mathfrak{m}_2 = -C_2\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

We now define the null basis vector $A_1 := p$, and extend to a basis (A_1, A_2, A_3) by defining, for example,

$$A_2 = e_2 + \frac{p^2}{p^3}e_1, \qquad A_3 = -\frac{1}{p^3}e_1.$$

We let

$$Q := A_3 \times v = \langle v, A_2\rangle A_3 - \langle v, A_3\rangle A_2,$$

and then define $g = (Q, A)$. Again it follows that $\eta = \mathrm{Ad}^*(g)\mu$ and therefore that the map $(p, v) \mapsto (g(p, v)^{-1}, (p, v))$ is a cross-section of the Marsden–Weinstein fibration.

The explicit parameterizations of the orbits given in Section 4.1 have the property that $\eta^*(\sigma^\mu) = \mathrm{d}t$, and hence $v^\mu = 1$. From the explicit forms of the cross-sections, g, it is straightforward to check for each type of orbit that $g(t)^{-1}g(t)' + g(t)^{-1}\mathcal{H}[\eta(t)]g(t)$ lies in $\mathfrak{g}_\mu$ for all t in the relevant range, as required. We may then, by direct integration, compute the gauge transformation (24). The integral curves of the Euler–Lagrange system are then given by (25).

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Appendix A. Pfaffian differential systems with one independent variable

Definition A.1. Let M be a smooth manifold. A *Pfaffian differential system* $(\mathcal{I}, \omega)$ with one independent variable on M consists of a Pfaffian differential ideal $\mathcal{I} \subset \Omega^*(M)$ and a non-vanishing 1-form $\omega \in \Omega^1(M)$ such that $\omega \not\equiv 0 \ (\mathrm{mod}\ \mathcal{I})$.

Definition A.2. An *integral element* of $(\mathcal{I}, \omega)$ is a pair (x, E) consisting of a point $x \in M$ and a one-dimensional linear subspace $E \subset T_x(M)$ such that $\eta|_E = 0, \forall \eta \in \mathcal{I}$ and $\omega|_E \neq 0$. We denote by $V(\mathcal{I}, \omega)$ the set of integral elements of $(\mathcal{I}, \omega)$. We say that $\mathcal{I}$ has constant rank if it is generated by the cross-sections of a sub-bundle Z of $T^*(M)$.

Definition A.3. A (*parameterized*) *integral curve* of $(\mathcal{I}, \omega)$ is a smooth curve $\alpha : (a, b) \subseteq \mathbb{R} \to M$ such that

$$\alpha^*(\eta) = 0 \quad \forall \eta \in \mathcal{I}, \qquad \gamma^*(\omega) = \mathrm{d}t.$$

We denote the set of integral curves of the system by $\mathcal{V}(\mathcal{I}, \omega)$.

Definition A.4. We say that the Pfaffian system in one independent variable $(\mathcal{I}, \omega)$ is *reducible* if there exists a non-empty submanifold $M^* \subseteq M$ such that:

- for each point $x \in M^*$ there exists an integral element $(x, E) \in V(\mathcal{I}, \omega)$ tangent to M^*;
- if $N \subseteq M$ is any other submanifold with the same property then $N \subseteq M^*$.

We call M^* the *reduced space*. We define on M^* the *reduced Pfaffian system*, denoted by $(\mathcal{I}^*, \omega)$, which is obtained by restricting the original system $(\mathcal{I}, \omega)$ to M^*.

A basic result is the following, a proof of which may be found in [12].

Proposition A.5. *The Pfaffian systems* $(\mathcal{I}, \omega)$ *and* $(\mathcal{I}^*, \omega)$ *have the same integral curves.*

There is an algorithmic procedure for constructing the reduction of a Pfaffian system [12]. To construct the reduced space M^*, we consider the projection $M_1 \subseteq M$ of $V(\mathcal{I}, \omega)$ to M. If M_1 is a non-empty submanifold of M, we then define $(\mathcal{I}_1, \omega_1)$ to be the restriction of $(\mathcal{I}, \omega)$ to M_1. We then construct $V(\mathcal{I}_1, \omega_1)$, the set of integral elements of $(\mathcal{I}_1, \omega_1)$. Repeating this construction, we inductively define

$$M_k = (M_{k-1})_1, \qquad \mathcal{I}_k = (\mathcal{I}_{k-1})_1, \qquad \omega_k = (\omega_{k-1})_1.$$

This process defines a sequence $M \supseteq M_1 \supseteq \cdots \supseteq M_k \supseteq \cdots$ of submanifolds of M. If $M^* := \bigcap_{k\in\mathbb{N}} M_k \neq \emptyset$ then M^* is the reduced space of the system. Notice that this procedure requires that, at each stage, the subset $M_k \subseteq M_{k-1}$ is a non-empty submanifold.

A.1. Cartan systems

Let $\Psi \in \Omega^2(M)$ be an exterior differential 2-form on M. We define the *Cartan ideal* to be the Pfaffian differential ideal $\mathcal{C}(\Psi) \subseteq \Omega^*(M)$ generated by the set of 1-forms $\eta_V := i_V\Psi$ obtained by contracting Ψ with vector fields on M. If $\theta^1, \ldots, \theta^n$ is a local coframing on M and if $\Psi = a_{ij}\theta^i \wedge \theta^j$, then $\mathcal{C}(\Psi)$ is locally generated by the 1-forms $a_{ij}\theta^j$. A *Cartan system* is a pair $(\mathcal{C}(\Psi), \omega)$ consisting of a Cartan ideal $\mathcal{C}(\Psi)$ and a 1-form $\omega \in \Omega^1(M)$ such that $\omega|_p \notin \mathcal{C}(\Psi)|_p$, for all $p \in M$.

Definition A.6. A Cartan system $(\mathcal{C}(\Psi), \omega)$ is *regular* if:

- it is reducible and the reduced phase space M^* is odd-dimensional;
- the 2-form $\Psi^* := \Psi|_{M^*} \in \Omega^2(M^*)$ is of maximal rank on M^*.

An important fact is that if $(\mathcal{C}(\Psi), \omega)$ is regular then the Cartan ideal $\mathcal{C}(\Psi^*)$ on M^* is the restriction to M^* of the Cartan ideal $\mathcal{C}(\Psi)$ on M:

$$\mathcal{I}^* := \mathcal{C}(\Psi)|_{M^*} = \mathcal{C}(\Psi^*).$$

Again, a proof of this result may be found in [12].

If $(\mathcal{C}(\Psi), \omega)$ is regular, then there exists a unique vector field ξ on M^* such that $i_\xi\Psi^* = 0$ and $\omega(\xi) = 1$. We call ξ the *characteristic vector field* of the Cartan system $(\mathcal{C}(\Psi), \omega)$. The integral curves of the characteristic vector field coincide with the parameterized integral curves of $(\mathcal{C}(\Psi^*), \omega)$, and hence with those of $(\mathcal{C}(\Psi), \omega)$.

A.2. Contact systems on jet spaces

Given a manifold M, we denote by $J^k(\mathbb{R}, M)$ the bundle of the k-order jets of maps $\gamma : \mathbb{R} \to M$. The k-jet of γ at t will be denoted by $j^k(\gamma)|_t$. Local coordinates $(x^1, \ldots, x^n)$ on M give standard local coordinates $(t, x^1, \ldots, x^n, x^1_1, \ldots, x^n_1, \ldots, x^1_k, \ldots, x^n_k)$ on the jet space $J^k(\mathbb{R}, M)$. With respect to such a coordinate system a k-jet with coordinates $(t, x^1, \ldots, x^n, x^1_1, \ldots, x^n_1, \ldots, x^1_k, \ldots, x^n_k)$ is represented by $j^k(\gamma)|_t$, where γ is the curve defined by

$$\gamma : s \mapsto (x^1, \ldots, x^n) + (x^1_1, \ldots, x^n_1)(t - s) + \cdots + \frac{1}{k!}(x^1_k, \ldots, x^n_k)(t - s)^k.$$

The *canonical contact system* $\mathcal{I}$ on $J^k(\mathbb{R}, M)$ is defined to be the Pfaffian differential ideal generated by the forms $\eta^i_a = \mathrm{d}x^i_a - x^i_{a+1}\,\mathrm{d}t$, $i = 0, \ldots, n$, $a = 0, \ldots, k-1$ (where $x^i_0 = x^i$). The independence condition of the system is given by the 1-form $\mathrm{d}t$. The integral curves $\Gamma : (a, b) \to J^k(\mathbb{R}, M)$ of $\mathcal{I}$ such that $\Gamma^*(\omega) = \mathrm{d}t$ are the canonical lifts $j^k(\gamma)$ of maps $\gamma : (a, b) \to M$.

Appendix B. Constrained variational problems in one independent variable

Definition B.1. Let $(\mathcal{I}, \omega)$ be a Pfaffian differential system on a smooth manifold M and let $L : M \to \mathbb{R}$ be a smooth function. The triple $(\mathcal{I}, \omega, L)$ is said to be a *constrained variational problem in one independent variable*. The function L is referred to as the *Lagrangian* of the variational problem.

The Lagrangian L gives rise to the action functional $\mathcal{L} : \mathcal{V}(\mathcal{I}, \omega) \to \mathbb{R}$ defined (perhaps not everywhere) on the space of the integral curves of $(\mathcal{I}, \omega)$ by

$$\mathcal{L}(\gamma) = \int_\gamma \gamma^*(L\omega).$$

Definition B.2. By an *extremal curve* of $(\mathcal{I}, \omega, L)$ we mean an integral curve γ that is a critical point of the functional $\mathcal{L}$ when one considers compactly supported variations of γ through integral curves of the system.

Let us suppose that the Pfaffian ideal $\mathcal{I}$ is generated by a sub-bundle $Z \subset T^*(M)$. We then let $\tilde{Z} \subset T^*(M)$ be the affine sub-bundle $L\omega + Z$. We denote by $\psi \in \Omega^1(\tilde{Z})$ the restriction to $\tilde{Z}$ of the tautological 1-form on $T^*(M)$, and call ψ the *Liouville form* of the variational problem. We let Ψ be the 2-form $\mathrm{d}\psi$ and we consider on $\tilde{Z}$ the Cartan system $\mathcal{C}(\Psi)$ together with the independence condition ω.

Definition B.3. We say that $(\mathcal{I}, \omega, L)$ is a *regular variational problem* if the Cartan system $(\mathcal{C}(\Psi), \omega)$ is reducible. The reduced space $Y \subset \tilde{Z}$ of $(\mathcal{C}(\Psi), \omega)$ is called the *momentum space* of the variational problem. The restriction of the Cartan system $(\mathcal{C}(\Psi), \omega)$ to Y is called the *Euler–Lagrange system* of the variational problem, and denoted $(\mathcal{E}, \omega)$.

The importance of the Euler–Lagrange system comes from the following theorem (cf. [3,12]).

Theorem B.4. *Let $\Gamma : (a, b) \to Y$ be an integral curve of the Euler–Lagrange system. Then $\gamma = \pi_M \circ \Gamma : (a, b) \to M$ is a critical point of the action functional $\mathcal{L}$, where $\pi_M : Y \to M$ denotes the restriction to Y of the projection $T^*(M) \to M$.*

This theorem allows us to find critical points of the variational problem from the integral curves the Euler–Lagrange system. However, not all the extremals arise this way for a general variational problem. It is known that if all the derived systems of Z have constant rank, then all the extremals are projections of the integral curves of the Euler–Lagrange system [3]. Other results in this direction have been proved by Hsu [16].

Definition B.5. A variational problem $(\mathcal{I}, \omega, L)$ is said to be *non-degenerate* if the Cartan system $(\mathcal{C}(\Psi), \omega)$ is regular, in the sense of Definition A.6.

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